Example 5

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad \begin{array}{ll} q = A\sin\omega t & (1) \\ p = m\omega A\cos\omega t & (2) \end{array}$$

$$Q = \omega t \qquad P = cA^n$$

We're looking for $F_1(q, Q, t)$, such that

$$\frac{\partial F_1}{\partial q} = p = m\omega A\cos\omega t = m\omega A\cos Q$$

from (1) $A = \frac{q}{\sin Q}$

$$\Rightarrow \frac{\partial F_1}{\partial q} = m\omega q \cot Q = p$$

$$-\frac{\partial F_1}{\partial Q} = P = c\left(\frac{q}{\sin Q}\right)^n$$

$$\frac{\partial^2 F_1}{\partial Q \partial q} = \frac{\partial^2 F_1}{\partial q \partial Q} \Rightarrow -m\omega q \frac{1}{\sin^2 Q} = -c\frac{nq^{n-1}}{\sin^n Q}$$

$$\Rightarrow n = 2 \quad \text{and} \quad -m\omega = -2c, \quad c = \frac{m\omega}{2}$$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial q} = m\omega q \cot Q = p \\ \frac{\partial F_1}{\partial Q} = -\frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q} \end{cases} \Rightarrow \underline{F_1 = \frac{m\omega q^2}{2}\cot Q}$$

$\Rightarrow$ Although cannonical transformations Q and P depended on t, F_1 doesn't depend.

$$K = H + \frac{\partial F_1}{\partial t} = H \quad \text{is total energy}$$

$$P = cA^2, \quad A = \sqrt{\frac{P}{c}} \quad \Rightarrow q = \sqrt{\frac{2P}{m\omega}} \sin Q$$

$$p = \sqrt{2Pm\omega} \cos Q$$

$$K_0 = \frac{2Pm\omega}{2m} \cos^2 Q + \frac{1}{2} m\omega^2 \frac{2P}{m\omega} \sin^2 Q =$$

$$= \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P$$

$$\Rightarrow \underbrace{E}_{\text{value of } K_0} = \frac{1}{2} \underbrace{k}_{m\omega^2} A^2$$

Because K_0 doesn't depend on $Q \Rightarrow P = \text{const}$

We could use these results also to find $q(t)$, and $p(t)$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega$$

$$Q = \omega t + \varphi_0$$

$$\Rightarrow q(t) = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \varphi_0) = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \varphi_0)$$

We have $F = F_1(q, Q, t)$

$$\frac{\partial F_1}{\partial q} = p \qquad \frac{\partial F_1}{\partial Q} = -P$$

Sometimes this doesn't work. So we use

$$F = F_2(q, P, t) - \sum Q_i P_i$$

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$\frac{dF}{dt} = -\sum \dot{Q}_i P_i - \sum Q_i \dot{P}_i + \sum \left(\frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i \right) + \frac{\partial F_2}{\partial t}$$

We require $p_i = \dfrac{\partial F_2}{\partial q_i}$ and $Q_i = \dfrac{\partial F_2}{\partial P_i}$

$$\Rightarrow H = K - \frac{\partial F_2}{\partial t}$$

Example 6

$F_2 = \sum_i q_i P_i \quad \Rightarrow \quad p_i = P_i, \quad Q_i = q_i$

and there is nothing new done

Let's chose $F_2 = \left(\sum_i f_i(\vec{q}) P_i \right) + g(\vec{q})$

(general form)

$Q_i = f_i(\vec{q})$ (transf-n in configuration space)

$$p_j = \left(\sum_i \frac{\partial f_i}{\partial q_j} P_i \right) + \frac{\partial g}{\partial q_j} \Rightarrow \vec{p} = \left(\frac{\partial f}{\partial q} \right) \left(\vec{P} \right) + \left(\frac{\partial g}{\partial q} \right)$$

Now we can redo all we have done, using matrix form.

(1) $\dot{q}_i = \frac{\partial H}{\partial p_i}$, (2) $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Introduce; $\vec{\eta} = \begin{pmatrix} q_1 \\ \vdots \\ q_K \\ p_1 \\ \vdots \\ p_K \end{pmatrix}$; $\frac{\partial H}{\partial \vec{\eta}} = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_K} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_K} \end{pmatrix}$

$$\Rightarrow \dot{\vec{\eta}} = \underbrace{\left(\begin{array}{c|c} 0 & \mathbb{1}_{K\times K} \\ \hline -\mathbb{1} & 0 \end{array}\right)}_{\substack{J \\ \downarrow \\ \text{symplectic matrix}}} \left(\frac{\partial H}{\partial \vec{\eta}}\right)$$

(Symplectic formulation)

Introduce; $\left(\vec{\zeta}(\vec{\eta})\right) = \begin{pmatrix} Q_1 \\ \vdots \\ Q_K \\ P_1 \\ \vdots \\ P_K \end{pmatrix}$

$$\left(\dot{\vec{\zeta}}\right) = (J)\left(\frac{\partial K}{\partial \vec{\zeta}}\right)$$

HEM for new hamiltonian $K(\vec{\zeta})$

Let's assume this transformation is not time dependent.

$$\left(\dot{\vec{\zeta}}\right) = \underbrace{\left(\frac{\partial \zeta_i}{\partial \eta_j}\right)}_{\text{Jakobian matrix } (M_{ij})} \left(\dot{\vec{\eta}}\right)$$

$$\Rightarrow \left(\dot{\vec{\zeta}}\right) = (M_{ij})(\mathcal{J})\left(\frac{\partial H}{\partial \vec{\eta}}\right)$$

As K is same as H (except some variables), we can write:

$$\left(\frac{\partial H}{\partial \vec{\eta}}\right) = \left(\frac{\partial \vec{\zeta}}{\partial \vec{\eta}}\right)\left(\frac{\partial K}{\partial \vec{\zeta}}\right)$$

$$\frac{\partial H}{\partial \eta_i} = \sum_j \frac{\partial \zeta_j}{\partial \eta_i} \frac{\partial K}{\partial \zeta_j}$$

$$\Rightarrow \left(\dot{\vec{\zeta}}\right) = (M_{ij})(\mathcal{J})(M_{ji})\left(\frac{\partial K}{\partial \vec{\zeta}}\right) =$$

$$= (M)(\mathcal{J})(M)^T\left(\frac{\partial K}{\partial \vec{\zeta}}\right)$$

So we can conclude:

$$(*) \quad \boxed{(\mathcal{J}) = (M)(\mathcal{J})(M)^T} \quad \left(\text{Symplectic condition}\right)$$

So now we can use this and check if transformation is cannonical.

If we take det of $(*) \Rightarrow \det(M) = 1$

$$\Rightarrow \underbrace{\iiint \cdots}_{(2k)} \underbrace{\left|\frac{\partial \vec{\zeta}}{\partial \vec{\eta}}\right|}_{|M|=1} d^{2k}\vec{\eta} = \underbrace{\iiint \cdots}_{(2k)} d^{2k}\vec{\zeta}$$

$\Rightarrow$ volume under cannonical tr.-n doesn't change

Assume we have a transf.-n of following kind;

$$\vec{\eta}(t_0) \to \underset{\substack{\downarrow \\ \text{short time} \\ \text{after } t_0}}{\vec{\eta}(t)} = \vec{\eta}(t_0) + dt\,\dot{\vec{\eta}} = \vec{\zeta}$$

(infinitessimal canonical transformation — ICT)

Now let's try to find gen. f.-n for this transf.-n.

We need to use identity tr.-n + smth. $\Rightarrow$

$$F_2 = \sum q_i P_i + dt\,G(\vec{q}, \vec{P}, t) \qquad p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial p_i}$$

$$p_i = P_i + dt\frac{\partial G}{\partial q_i}, \qquad Q_i = q_i + dt\frac{\partial G}{\partial P_i}$$

$$P_i = p_i + dt\,\dot{p}_i$$

$$\dot{p}_i = -\frac{\partial G}{\partial q_i}\bigg|_{\vec{q},\,\vec{P}=\vec{p}} \qquad \dot{q}_i = \frac{\partial G}{\partial P_i}\bigg|_{\vec{q},\,\vec{P}=\vec{p}}$$

$$\left(\dot{\vec{\eta}}\right) = (J)\left(\frac{\partial G}{\partial \vec{\eta}}\right) \Big/ \times dt \Rightarrow$$

(small change from $\frac{\partial G}{\partial ..}(\vec{q}, \vec{P})$ 2nd order in dt → can neglect)

$$\Rightarrow \delta(\vec{\eta}) = (J)\left(\frac{\partial G}{\partial \vec{\eta}}\right)dt$$

$$\underbrace{\frac{\partial \vec{\zeta}}{\partial \vec{\eta}} = 1 + dt(J)}_{(M)}\left(\frac{\partial^2 G}{\partial \eta_i \partial \eta_j}\right); \quad (M)^T = 1 - dt\underbrace{\left(\frac{\partial^2 G}{\partial \eta_j \partial \eta_i}\right)}_{\text{same}}\cdot(J)$$

$$(M)^T(J)(M) = (J) + (J)dt(J)\left(\frac{\partial^2 G}{\partial \eta_i \partial \eta_j}\right) -$$

$$- dt\left(\frac{\partial^2 G}{\partial \eta_j \partial \eta_i}\right)(J)(J)\mathbb{1} = (J)$$

$\Rightarrow$ volume in phase space also doesn't change