3/30/06 Poisson Brackets Continued

\[ \{ u, v \} = -[v, u] \]

\[ u(t, x) \quad v(t, x) \]

\[ [a u_1 + b u_2, v] = a[u_1, v] + b[u_2, v] \]

\[ [u_1, u_2, v] = u_1[u_2, v] + u_2[u_1, v] \]

\[ u, [v, w] + [v, [w, u]] + [w, [u, v]] = 0 \]

Proof left as an exercise for the reader.

\[ [u, v] = \left( \frac{\partial u}{\partial \dot{q}} \right)^T \mathbf{J} \left( \frac{\partial v}{\partial \dot{q}} \right), \quad \text{so} \]

\[ [u, [v, w]] = \left( \frac{\partial u}{\partial \dot{q}} \right)^T \mathbf{J} \left( \frac{\partial}{\partial \dot{q}} \left( \mathbf{J} \left( \frac{\partial v}{\partial \dot{q}} \right)^T \mathbf{J} \left( \frac{\partial w}{\partial \dot{q}} \right) \right) \right) \]

\[ \dot{q}_M - [u, v] \rightarrow \frac{1}{\hbar} \left[ \dot{q}, \dot{p} \right] \]

Refomulating Classical Mechanics in the Poisson Bracket Formalism.

Recall: \( \dot{q} + \dot{\dot{q}} = \mathbf{J} \)

\[ F = \sum g_i \dot{q}_i + \epsilon G (\mathbf{q}, \mathbf{p}, t), \quad \text{so} \]

\[ \dot{\dot{q}} ^\mathbf{c} = \epsilon \mathbf{J} \left( \frac{\partial G}{\partial \dot{q}} \right) \]

which implies

\[ \dot{\dot{q}} ^\mathbf{c} = \epsilon [\dot{q}, G] \]

since

\[ [[\dot{q}, G]] = \left( \frac{\partial \dot{q}}{\partial \dot{q}} \right)^T \mathbf{J} \left( \frac{\partial G}{\partial \dot{q}} \right) \]
\[
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

Example: \(G = H\)

\[
\frac{\partial^2 \psi}{\partial \tau^2} = \frac{\partial}{\partial \tau} \left[ \frac{\partial \psi}{\partial \tau} \right] = \frac{\partial}{\partial \tau} \left[ \psi, H \right]
\]

\[
u(t) \rightarrow \nu + \delta \nu ; \quad \delta \nu = \left( \frac{\partial \nu}{\partial \psi} \right)^T \delta \psi
\]

\[
= \epsilon \left( \frac{\partial \nu}{\partial \psi} \right)^T \left( \frac{\partial G}{\partial \psi} \right)
\]

\[
= \epsilon [\nu, G]
\]

If \(\nu\) is not explicitly dependent on time,

\[
u = [\nu, H] + \frac{\partial \nu}{\partial t} = [\nu, H] + \frac{\partial}{\partial t}
\]

So, if we are interested in small variations in energy (and it has no time dependence), then

\[
[\psi, H] = \epsilon [H, G]
\]

Emerging Noether: If a change in \(\nu, G\) leaves system invariant (\(\delta H = 0\)), then \(G\) is conserved.
Recall $g_i \rightarrow \partial_i \Phi$, then $G = \Phi \Rightarrow$ if $H$

is invariant under translation in $i$-direction $\Rightarrow$ Poincaré

Once we discover two conserved quantities, we can find more...

Suppose $G_1, G_2$ are conserved and $\partial G_1 = \partial G_2 = 0$, then

$[G_1, H] = 0; \quad [G_2, H] = 0$

and

$[H, [G_1, G_2]] = -[G_1, [G_2, H]] - [G_2, [H, G]]$

which implies $[G_1, G_2]$ is invariant. (see HW PS #9)

If we know $u(t)$, where $u(0) = u_0$ for any

system, where $u$ contains combinations of

position & momentum, then the system is

essentially solved non-uniquely, let $x \in \text{IR}$. Then

for a specific transformation given by $x \mapsto x' \mapsto x$

and

$\delta x = d \alpha \int \left( \frac{\partial G}{\partial \dot{x}} \right) \delta x$

so, expand $u$ near $\alpha$

$u(x) = u_0 + \frac{du}{d\alpha} \alpha + \frac{1}{2} \frac{d^2 u}{d\alpha^2} \alpha^2 + \cdots$

$u(x) = u_0 + \left[ u, G \right] \alpha + \frac{1}{2} \left[ [u, G], G \right] \alpha^2 + \cdots$

Example: Marble in Free Fall

$H = \frac{p^2}{2m} + m g y$;

$y(t) = y_0 + \left[ g, H \right] t - \frac{1}{2} \left[ g, \left[ g, H \right] \right] t^2$

$\frac{H}{\partial p} = \frac{g}{m} \Rightarrow \text{Huebrackets are zero}$

$y(t) = y_0 + \frac{\dot{y}}{\dot{m}} t - \frac{1}{2} \frac{g \dot{t}^2}{m} \Rightarrow \text{Q.E.D.}$
Example: What is the rotation generator?

Suppose we have 6 d.o.f. \( \Rightarrow \) 6 generalized coordinates \((x, y, z, p_x, p_y, p_z)\), and we rotate by \( \theta \):

\[
\begin{align*}
x & \rightarrow x + y \, d\theta \\
y & \rightarrow y + x \, d\theta
\end{align*}
\]

So,

\[
\delta \dot{q}_i = \frac{\partial \theta}{\partial \tilde{q}_i} \frac{\partial \tilde{q}_i}{\partial \theta}
\]

and

\[
\begin{align*}
\delta x &= \frac{\partial \theta}{\partial p_x} \delta p_x \\
\delta y &= \frac{\partial \theta}{\partial p_y} \delta p_y \\
\delta p_x &= -\frac{\partial \theta}{\partial x} \delta x \\
\delta p_y &= -\frac{\partial \theta}{\partial y} \delta y
\end{align*}
\]

Therefore

\[
\begin{align*}
y &= \frac{\partial \tilde{g}}{\partial p_x} ;
 x &= \frac{\partial \tilde{g}}{\partial p_y}
\end{align*}
\]

Hence,

\[
G = x \frac{\partial \tilde{g}}{\partial p_y} - y \frac{\partial \tilde{g}}{\partial p_x} = L_z
\]

Applying what we've recently learned (I.C.T.)

\[
u \rightarrow \nu_{rot} = u + \delta u \Rightarrow \delta u = [u, L_z] \theta
\]

For an arbitrary axis \( \hat{n} \), rotation by \( \theta \) \( \Rightarrow \tilde{\hat{n}} \cdot \hat{n} \).

So for a small rotation, and \( \mathbf{F} = \mathbf{F}_0 \mathbf{a}_0 \mathbf{p}_i \)

\[
\delta \mathbf{F} = \delta \mathbf{F} = \mathbf{F}_0 \delta \theta \mathbf{a}_0 \mathbf{p}_i \Rightarrow \delta \mathbf{F} = \mathbf{F}_0 \theta \mathbf{a}_0 \mathbf{p}_i \times \mathbf{F},
\]

so

\[
[\mathbf{F}, \mathbf{L} \cdot \hat{n}] = \hat{n} \times \mathbf{F} \Rightarrow [\mathbf{L}, \mathbf{L} \cdot \hat{n}] = \hat{n} \times \mathbf{L}
\]

For example,

\[
[L_x, L_y] = (\hat{n} \times \mathbf{L})_x = L_z
\]

\[
[L^2, \mathbf{L} \cdot \hat{n}] = 0 \text{ since rotation of any scalar}
\]

like \( L^2 \) leaves it unchanged.