02.16.06

Classical Mechanics

$$\vec{\omega} = \begin{pmatrix} \cos\psi\,\dot\theta + \sin\psi\sin\theta\,\dot\phi \\ -\sin\psi\,\dot\theta + \cos\psi\sin\theta\,\dot\phi \\ \dot\psi + \cos\theta\,\dot\phi \end{pmatrix}$$

$$I_1 = I_2 = I_\perp \, ; \, I_3$$

$$T = \frac{1}{2} I_\perp(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

$$\mathcal{L} = \frac{1}{2} I_\perp\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2} I_3(\dot\psi + \cos\theta\,\dot\phi)^2 - V(\theta)$$

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = I_3 \omega_3 \text{ const.}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = I_\perp \sin^2\theta\,\dot\phi + I_3\omega_3\cos\theta \text{ const.}$$

$$\underset{\text{const.}}{E} = \frac{1}{2} I_\perp\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2} I_3\omega_3^2 + V(\theta)$$

$$E' = E - \frac{1}{2} I_3\omega_3^2 = \frac{1}{2} I_\perp \dot\theta^2 + V(\theta) = \frac{1}{2} I_\perp \dot\theta^2 + \underbrace{\frac{1}{2}\frac{(p_\phi - I_3\cos\theta\,\omega_3)^2}{I_\perp \sin^2\theta} + V(\theta)}_{V'(\theta)}$$

$$E' = \frac{1}{2} I_\perp \dot\theta^2 + V'(\theta)$$

$$\frac{p_\phi - I_3\omega_3\cos\theta}{I_\perp \sin^2\theta} = \dot\phi$$

$$\frac{1}{2} \cancel{I_\perp} \sin^2\theta \; \frac{(p_\phi - I_3\omega_3\cos\theta)^2}{I_\perp^2 \sin^2\theta}$$

$$I_\perp \ddot\theta = -\frac{\partial V'}{\partial \theta}$$

$$\ddot\theta = -\frac{1}{I_\perp}\frac{\partial V'}{\partial \theta}$$

$$I_\perp \ddot{\theta} = -\frac{\partial V'}{\partial \theta}$$

$$= -\frac{1}{\cancel{2} I_\perp} \frac{\cancel{2}(P_\phi - I_3 \omega_3 \cos\theta)\, I_3 \sin^3\theta\, \omega_3 - (P_\phi - I_3 \cos\theta\, \omega_3)^2\, 2\sin\theta \cos\theta}{\sin^{\cancel{4}\,3}\theta} - \frac{\partial V}{\partial \theta}$$

$$= \cancel{-\frac{1}{2I_\perp}}$$

$$= -\dot{\phi}\, \frac{I_3 \sin^2\theta\, \omega_3 - P_\phi \cos\theta + I_3 \cos^2\theta\, \omega_3}{\sin\theta} - \frac{\partial V}{\partial \theta}$$

$$= \frac{P_\phi \cos\theta - I_3 \omega_3}{\sin\theta}\, \dot{\phi} - \frac{\partial V}{\partial \theta}$$

$$\ddot{\theta} = \frac{\dot{\phi}}{I_\perp} \frac{P_\phi \cos\theta - I_3 \omega_3}{\sin\theta} - \frac{1}{I_\perp} \frac{\partial V}{\partial \theta}$$

1) freely rotating object – when is $\theta = $ const? $\left(\frac{\partial V}{\partial \theta} = 0\right)$

i) $\dot{\phi} = 0 \Rightarrow \ddot{\theta} = 0$; θ, ϕ are constant → rotating around fixed axis

ii) $P_\phi \cos\theta - I_3 \omega_3 = 0$

$= I_\perp \sin^2\theta \cos\theta\, \dot{\phi} + I_3 \omega_3 \cos^2\theta - I_3 \omega_3 = 0$

$= \sin^2\theta \left(I_\perp \cos\theta\, \dot{\phi} - I_3 \omega_3 \right)$

(iii) α) $\theta = 0$ → again, fixed axis ($\dot{\phi}$ and $\dot{\psi}$ are indistinguishable and their sum is constant)

β) $I_\perp \cos\theta\, \dot{\phi} = I_3 \omega_3$ → can be solved for θ, given $\dot{\phi}, \omega_3$

⇒ precession around body cone @ constant $\theta, \dot{\phi}, \omega_3$

2)

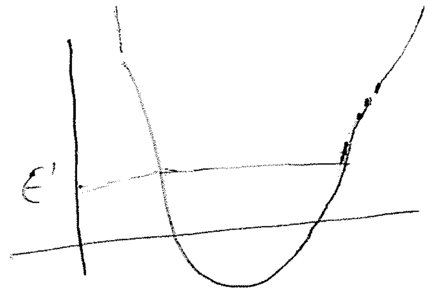
$$V(\theta) = mgl \cos\theta$$

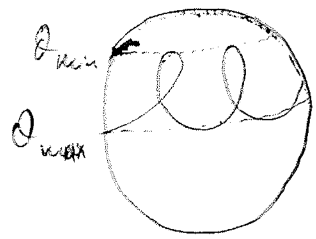
$$\ddot{\theta} = \dot{\phi} \sin\theta \left(\cos\theta\, \dot{\phi} - \frac{I_3}{I_\perp} \omega_3 \right) + \frac{mgl}{I_\perp} \sin\theta$$

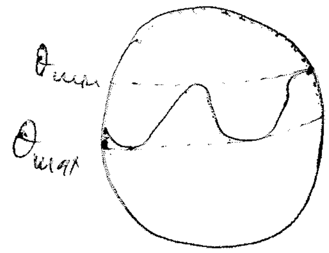
$E' = \frac{1}{2} I_\perp \dot{\theta}^2 + V'(\theta)$

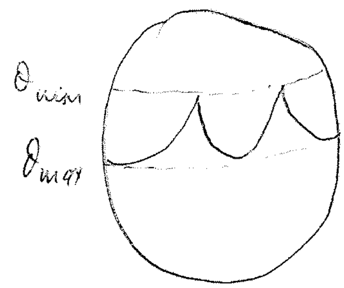
$E' - V'(\theta) = 0$

maxima + minima of θ









$\ddot{\theta} = 0$ i) $\dot{\phi}\left(\cos\theta\,\dot{\phi} - \frac{I_3}{I_\perp}\omega_3\right) + \frac{mgl}{I_\perp} = 0$

$\theta = \pi/2$ requires $-\dot{\phi}\,\omega_3 \frac{I_3}{I_\perp} + \frac{mgl}{I_\perp} = 0$

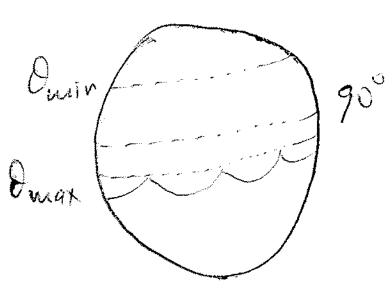
$\dot{\psi}\dot{\phi} I_3 = mgl$ so $\dot{\phi}$ cannot be zero initially

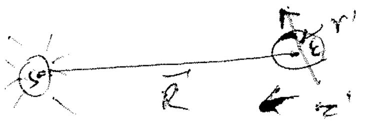
ii) initial condition

$\dot{\psi} = \dot{\psi}_0$ large

$\dot{\phi} = 0$ $\theta = 90^\circ \rightarrow E' = 0$

$\dot{\theta} = \frac{mgl}{I_\perp}$





$$V = \iiint -GM \frac{\rho \, d^3\vec{r}\,'}{|\vec{R} - \vec{r}\,'|}$$

$$= -\frac{GM}{R} \iiint \frac{\rho \, d^3\vec{r}\,'}{\left|1 - \frac{r'}{R}\cos\theta\right|}$$

$$\frac{1}{\left|1 - \frac{r'}{R}\cos\theta\right|} = \sum_{l=0}^{\infty} \frac{r'^l}{R^l} P_l(\cos\theta)$$

$$V = -\frac{GMR}{R} - \frac{GM}{R^3} \iiint \rho \, \vec{r}\,'^2 d^3\vec{r}\,' \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$\frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{3}{2}(1 - \sin^2\theta) - \frac{1}{2} = 1 - \underbrace{\frac{3}{2}\sin^2\theta}_{-\frac{3}{2}\,\cancel{\sin}\,\hat{R}\, I\, \hat{R}}$$

$$r'^2 = x'^2 + y'^2 + z'^2$$

$$\begin{pmatrix} x^2 + y^2 \\ y^2 - z^2 \\ z'^2 + x'^2 \end{pmatrix} \quad \frac{1}{2}\theta \quad \frac{1}{2}(I_1 + I_2 + I_3) \;\Big|\; \frac{3}{2}(I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2)$$

$$\frac{1}{2}\,\mathrm{tr}(\mathbb{I}) - \frac{3}{2}\,\hat{R}\cdot\mathbb{I}\cdot\hat{R}$$