**Graduate Quantum Mechanics II - Problem Set 5**

**Problem 1)**

Given the well-known fact that for a classical bound state orbit in a 1/r potential (Coulomb or gravitational) the relationship between the total (binding) energy and the average kinetic and potential energy is given by $|E| = \langle T_{\text{kin}} \rangle = \frac{1}{2} \langle V \rangle$, express the radius for a classical circular orbit in terms of the angular momentum $L$ and the binding energy $E$ of the orbit. Compare with the most probable value for $r$ if measured on a quantum mechanical eigenstate of the same potential with very large $\ell = m$ and main quantum number $n = \ell + 1$ (see lecture notes). Comment?

SOLUTION: Mostly simple. The comment is that the classical radius and the most probably radius for the correct quantum mechanical solution are the same.

**Problem 2)**

Calculate the energy difference between the qu.m. state defined above in Problem 1 and the state directly below it, with quantum numbers $n - 1, \ell - 1$ and $m - 1$. Express your result in terms of $E_n$ and $\ell$.

Find the angular velocity of a classical particle in a circular orbit (1/r potential) in terms of the same quantities, $E$ and $L$. Compare your results. Comment?

SOLUTION: The answer for $\Delta E$ comes out as $\Delta E = E_n \frac{2\ell + 1}{\ell^2} = \frac{2E_n}{\ell}$ which means that the frequency of the emitted light must be $\omega = \frac{\Delta E}{\hbar} = \frac{2E_n}{\hbar\ell} = \frac{2E_n}{L_z}$. This is the same as the classical angular velocity, and therefore the frequency of the dipole radiation expected.

**Problem 3)**

Write down the full time-dependent wave function (in coordinate space) of a particle in a one-dimensional constant potential of magnitude $V_0$. Assume the particle is in an (“improper”) eigenstate of the Hamiltonian. What is the phase velocity of the corresponding de-Broglie wave? If instead I had a superposition of waves (over a narrow range of energies) forming a traveling “wave packet”, what would be the group velocity (the velocity of the amplitude “envelope”) of that packet? Could the two ever be equal?

SOLUTION: Not difficult; just keep in mind that the phase velocity is $\omega/k = E/p$ and the group velocity is $\partial \omega / \partial k = \partial E / \partial p$ where $E = p^2/2m + V_0$. 
**Problem 4)**

Evaluate the Wigner function, \( W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' e^{-ipx' / \hbar} \psi^*(x - \frac{x'}{2}) \psi(x + \frac{x'}{2}) \), explicitly for the 1-dimensional Gaussian wave packet \( \psi(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{ipx / \hbar} e^{-x^2 / 4\sigma^2} \) and show that it indeed describes the “joint probability density” for momentum \( p \) and position \( x \).

**SOLUTION:**

Plugging in the wave function in the definition of the Wigner function, we get

\[
W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' e^{-ipx' / \hbar} \frac{1}{\sqrt{2\pi \sigma}} e^{-ip(x-x'/2)/\hbar} e^{-x'(x-x'/2)^2 / 4\sigma^2} \frac{1}{\sqrt{2\pi \sigma}} e^{ip(x+x'/2)/\hbar} e^{-x'(x+x'/2)^2 / 4\sigma^2}
\]

\[
= \frac{1}{2\pi\hbar} \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} dx' e^{-ipx' / \hbar} e^{-ip(x-x'/2)/\hbar} e^{ip(x-x'/2)/\hbar} e^{-x'(x-x'/2)^2 / 4\sigma^2} e^{-x'(x+x'/2)^2 / 4\sigma^2}
\]

\[
= \frac{1}{2\pi\hbar} \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} dx' e^{-i(px-p_0)x'/\hbar} e^{-x'^2 / 2\sigma^2} e^{-x^2 / 8\sigma^2} \frac{1}{\sqrt{2\pi \sigma}} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' e^{-i(px-p_0)x'/\hbar} e^{-x'^2 / 8\sigma^2}
\]

The last integral is just the Fourier Transform of a Gaussian. Following Shankar’s appendix, this becomes

\[
\int_{-\infty}^{\infty} dx' e^{-i(px-p_0)x'/\hbar} e^{-x'^2 / 8\sigma^2} = \sqrt{8\pi \sigma^2} e^{-\frac{(p-p_0)^2}{2h^2\sigma^2}}
\]

Plugging it in yields

\[
W(x, p) = e^{-x'^2 / 2\sigma^2} \frac{\sqrt{8\pi \sigma^2}}{2\pi\hbar} e^{-\frac{(p-p_0)^2}{2h^2\sigma^2}} = e^{-x'^2 / 2\sigma^2} \frac{1}{\sqrt{2\pi \sigma \hbar / 2\sigma}} e^{-\frac{(p-p_0)^2}{2h^2\sigma^2}}
\]

which is indeed the product of two Gaussians, one in \( x \) with width \( \sigma \), the other in \( p \) centered on \( p_0 \) with width \( h/2\sigma \). Not only does this agree with our knowledge of the probability distributions both in position and in momentum for the wave packet, but it also shows that in fact there is no correlation between \( x \) and \( p \) – the momentum distribution is the same at each \( x \) and vice versa.