Graduate Quantum Mechanics – Final Exam - Solution

Problem 1)

We know that the momentum operator in one dimension can be written in x-basis as follows:

$$\langle x|\mathbf{P}|\psi\rangle = \frac{\hbar}{i}\frac{\partial\psi(x)}{\partial x} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle \ .$$

a) Show that, similarly, the position operator in momentum basis can be written as $\langle p|\mathbf{X}|\psi\rangle = i\hbar\frac{\partial\tilde{\psi}(p)}{\partial p} \equiv i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$, where

$$\tilde{\psi}(p) \equiv \langle p|\psi\rangle = \int_{-\infty}^{\infty} dx\,\langle p|x\rangle\langle x|\psi\rangle; \quad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}$$

is the Fourier transform of $\psi(x) \equiv \langle x|\psi\rangle$.

Hint: Plug in $\mathbf{X} = \int_{-\infty}^{\infty} dx\,|x\rangle x\langle x|$ and show that both sides are indeed equal.

Answer: To prove: $\langle p|\mathbf{X}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$. Left hand side (using hint):

$\langle p|\mathbf{X}|\psi\rangle = \int_{-\infty}^{\infty} dx\,\langle p|x\rangle x\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}x\psi(x)$. Right hand side:

$$i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle = i\hbar\frac{\partial}{\partial p}\int_{-\infty}^{\infty} dx\,\langle p|x\rangle\langle x|\psi\rangle = i\hbar\frac{\partial}{\partial p}\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\psi(x)$$

$= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, i\hbar\frac{\partial}{\partial p}e^{-ipx/\hbar}\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, i\hbar(-ix/\hbar)e^{-ipx/\hbar}\psi(x)$ q.e.d.

b) Find the eigenfunctions in **momentum basis** for the following 1-D Hamiltonian: $\mathbf{H} = \frac{\mathbf{P}^2}{2m} + mg\mathbf{X}$! (Don't worry about the overall normalization!)

Answer: In momentum basis, the Schrödinger equation reads $\frac{p^2}{2m}\tilde{\psi}(p) + i\hbar gm\frac{\partial\tilde{\psi}(p)}{\partial p} = E\tilde{\psi}(p)$.

Since the wave function depends only on p, the partial differential can be replaced by a full differential, and we can write this equation as follows:

$$i\hbar gm\frac{d\tilde{\psi}(p)}{\tilde{\psi}(p)} = \left(E - \frac{p^2}{2m}\right)dp \Rightarrow \quad \int_{\tilde{\psi}(0)}^{\tilde{\psi}(p)}\frac{d\tilde{\psi}(p')}{\tilde{\psi}(p')} = \frac{1}{i\hbar gm}\int_0^p\left(E - \frac{p'^2}{2m}\right)dp' \Rightarrow$$

$$\ln\left(\frac{\tilde{\psi}(p)}{\tilde{\psi}(0)}\right) = \frac{-i}{\hbar gm}\left(Ep - \frac{p^3}{6m}\right) \Rightarrow \quad \tilde{\psi}(p) = \tilde{\psi}(0)\exp\left(-\frac{i}{\hbar gm}\left(Ep - \frac{p^3}{6m}\right)\right)$$

Problem 2)

Consider the two-dimensional harmonic oscillator with Hamiltonian

$$\mathbf{H} = \frac{\mathbf{P}_x^2 + \mathbf{P}_y^2}{2m} + \frac{m\omega^2}{2}\left(\mathbf{X}^2 + \mathbf{Y}^2\right) = \hbar\omega\left[\frac{\hat{\mathbf{P}}_x^2 + \hat{\mathbf{P}}_y^2}{2} + \frac{\hat{\mathbf{X}}^2 + \hat{\mathbf{Y}}^2}{2}\right].$$

with $\hat{\mathbf{X}} = \sqrt{\frac{m\omega}{\hbar}}\mathbf{X}, \quad \hat{\mathbf{P}}_x = \sqrt{\frac{1}{m\omega\hbar}}\mathbf{P}_x$ etc.

a) Show that for each pair of eigenstates $|n_x\rangle, |n_y\rangle$ of eigenstates of the 1-dimensional harmonic oscillator Hamiltonian, with $\hbar\omega\left[\frac{\hat{\mathbf{P}}_x^2}{2} + \frac{\hat{\mathbf{X}}^2}{2}\right]|n_x\rangle = \left(n_x + \tfrac{1}{2}\right)\hbar\omega|n_x\rangle$, $\left[\frac{\hat{\mathbf{P}}_y^2}{2} + \frac{\hat{\mathbf{Y}}^2}{2}\right]|n_y\rangle = \left(n_y + \tfrac{1}{2}\right)|n_y\rangle$, the direct product $|n_x\rangle \otimes |n_y\rangle$ is an eigenstate of the 2-D hamiltonian **H**. What are the corresponding eigenvalues and what is the degree of degeneracy for each of these eigenvalues? *Note*: The operators $\mathbf{P}_x$ and **X** act only on the first factor in this product while being represented by a unit matrix for the second factor – and vice versa.

Answer:

$$\mathbf{H}\left\{|n_x\rangle \otimes |n_y\rangle\right\} = \left\{\hbar\omega\left[\frac{\hat{\mathbf{P}}_x^2}{2} + \frac{\hat{\mathbf{X}}^2}{2}\right]|n_x\rangle\right\} \otimes \mathbf{1}|n_y\rangle + \mathbf{1}|n_x\rangle \otimes \left\{\hbar\omega\left[\frac{\hat{\mathbf{P}}_y^2}{2} + \frac{\hat{\mathbf{Y}}^2}{2}\right]|n_y\rangle\right\}$$

$$= \hbar\omega\left\{\left(n_x + \tfrac{1}{2}\right)|n_x\rangle \otimes |n_y\rangle + \left(n_y + \tfrac{1}{2}\right)|n_x\rangle \otimes |n_y\rangle\right\} = \hbar\omega\left(n_x + n_y + 1\right)|n_x\rangle \otimes |n_y\rangle$$

so $|n_x\rangle \otimes |n_y\rangle$ is an eigenvector with eigenvalue (N+1), where $N = n_x + n_y$. For a given eigenvalue (i.e., for a given N), the degree of degeneracy is simply equal to the number of possible combinations n_x, n_y with $N = n_x + n_y$. Since both n_x, n_y can be zero or larger, this is simply equal to N+1. This means that the ground state (N = 0) is unique, the first excited state is 2-fold degenerate, etc. (This degeneracy corresponds to different allowed values of L^z.)

b) The angular momentum operator for rotations around the z-axis is given by $\mathbf{L}_z = \mathbf{X}\mathbf{P}_y - \mathbf{Y}\mathbf{P}_x = \hbar\left(\hat{\mathbf{X}}\hat{\mathbf{P}}_y - \hat{\mathbf{Y}}\hat{\mathbf{P}}_x\right) \equiv \hbar\hat{\mathbf{L}}_z$.

Show that it commutes with the Hamiltonian and therefore, that it must be possible to construct a basis of joint eigenstates to **H** and $\mathbf{L}_z$.

Answer: $\left[\mathbf{H}, \mathbf{L}_z\right] = \hbar\omega^2\left[\frac{\hat{\mathbf{P}}_x^2 + \hat{\mathbf{P}}_y^2}{2} + \frac{\hat{\mathbf{X}}^2 + \hat{\mathbf{Y}}^2}{2}, \hat{\mathbf{X}}\hat{\mathbf{P}}_y - \hat{\mathbf{Y}}\hat{\mathbf{P}}_x\right]$. Keeping only terms for which we know that the commutators are not zero, we find

$$\frac{\left[\mathbf{H}, \mathbf{L}_z\right]}{\hbar\omega^2} = \left[\frac{\hat{\mathbf{P}}_x^2}{2}, \hat{\mathbf{X}}\hat{\mathbf{P}}_y\right] - \left[\frac{\hat{\mathbf{P}}_y^2}{2}, \hat{\mathbf{Y}}\hat{\mathbf{P}}_x\right] - \left[\frac{\hat{\mathbf{X}}^2}{2}, \hat{\mathbf{Y}}\hat{\mathbf{P}}_x\right] + \left[\frac{\hat{\mathbf{Y}}^2}{2}, \hat{\mathbf{X}}\hat{\mathbf{P}}_y\right] =$$

$$= \frac{1}{2}\left\{\hat{\mathbf{P}}_x\left[\hat{\mathbf{P}}_x, \hat{\mathbf{X}}\right]\hat{\mathbf{P}}_y + \left[\hat{\mathbf{P}}_x, \hat{\mathbf{X}}\right]\hat{\mathbf{P}}_x\hat{\mathbf{P}}_y - \hat{\mathbf{P}}_y\left[\hat{\mathbf{P}}_y, \hat{\mathbf{Y}}\right]\hat{\mathbf{P}}_x - \left[\hat{\mathbf{P}}_y, \hat{\mathbf{Y}}\right]\hat{\mathbf{P}}_y\hat{\mathbf{P}}_x - \ldots + \ldots\right\}$$

$$= \frac{1}{2}\left\{-2i\hat{\mathbf{P}}_x\hat{\mathbf{P}}_y + 2i\hat{\mathbf{P}}_y\hat{\mathbf{P}}_x - 2i\hat{\mathbf{Y}}\hat{\mathbf{X}} + 2i\hat{\mathbf{X}}\hat{\mathbf{Y}}\right\} = 0, \text{ q.e.d.}$$

Problem 3)

I A deuterium atom, consisting of a Spin-1 nucleus (d=p+n) and a Spin-1/2 electron (e), is in its spatial ground state (n = 1, l = 0). When immersed in a very large magnetic field **B** pointing in z-direction (many Tesla), both the nucleus spin and the electron spin will (anti-)align themselves with the z-axis: M_s(d) = +1, M_s(e) = -1/2; therefore, the 2-particle spin state will be $\left|S_d = 1, M_s(d) = +1\right\rangle \otimes \left|S_e = \tfrac{1}{2}, M_s(e) = -\tfrac{1}{2}\right\rangle$. However, when the magnetic field gets turned off, the atom's spin wave function will collapse into an eigenstate of the total spin S_{d+e} and projection along the z-axis M_s(d+e) (due to hyperfine interaction).

a) What are the possible states | S_{d+e} , M_s(d+e)> the atom can collapse into? (List all possibilities).
Answer: According to the rules of angular momentum addition, the total spin can be either 3/2 or ½. The total list of all possible states is
$\left|S,M\right\rangle = \left\{\left|3/2,3/2\right\rangle, \left|3/2,1/2\right\rangle, \left|3/2,-1/2\right\rangle, \left|3/2,-3/2\right\rangle, \left|1/2,1/2\right\rangle, \left|1/2,-1/2\right\rangle\right\}$
However, because the initial state has M_s(d+e) = M_s(d) + M_s(e) = 1-1/2 = ½, only the 2nd and 5th entry in this list have non-zero overlap (probability) with the initial state.

b) What are the probabilities to find it in each of these states?
Answer: To find the probability for any one of these final states, we simply have to square the overlap with the initial state:
$\mathrm{Prob}(S,M) = \left|\left\langle S,M \middle| 1,1;1/2,-1/2\right\rangle\right|^2$. These are just the Clebsch-Gordan coefficients squared.
Therefore,
$$\mathrm{Prob}(3/2,1/2) = \left|\left\langle 3/2,1/2 \middle| 1,1;1/2,-1/2\right\rangle\right|^2 = \frac{1}{3}$$
$$\mathrm{Prob}(1/2,1/2) = \left|\left\langle 1/2,1/2 \middle| 1,1;1/2,-1/2\right\rangle\right|^2 = \frac{2}{3}$$
and all others are zero.

Problem 4)

This problem consist of a set of "trivia questions" which you should answer with no more than a few sentences each:

a) I measure observable **O** on some arbitrary state |ψ>, with the result of the measurement =o_0. What do I measure if I repeat the measurement immediately? Do I get the same result if I first wait a while and then repeat the measurement? Under what circumstance is the answer to this question "yes"?
Answer: After the first measurement, the wave function is in an eigenstate $\left|o_0\right\rangle$ of **O** with eigenvalue o_0, so after immediate re-measurement, I find the same result. If I wait a while, the wave function will evolve according to $\left|\psi\right\rangle(t) = e^{-i\mathbf{H}t/\hbar}\left|o_0\right\rangle$ which may no longer be an eigenstate of **O** (or not with the same eigenvalue). However, if the hamiltonian and **O** commute, the wave

function will remain in the same eigenstate of **O** and even after a while, the measurement will still yield o_0.

b) The Hamiltonian for a particle with spin **S** and magnetic moment $\boldsymbol{\mu}=\gamma\mathbf{S}$ in a magnetic field B along the z-asxis is given by $H = -\gamma\mathbf{S}_z B$. From what you know about generators of rotation, which components ($\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_z$) of the spin operator have expectation values (for any wave function) that are conserved under this Hamiltonian?

Answer: Since the three components of the spin operator do not commute with each others, only $\mathbf{S}_z$ commutes with the Hamiltonian. This means that $\mathbf{S}_z$ is conserved while $\mathbf{S}_x$ and $\mathbf{S}_y$ aren't.

c) A hydrogen atom is in the 2p (n=2, ℓ=1, k=0) eigenstate of the (spatial) Hamiltonian (in the following, ignore the electron and proton spin!). If we send it through a Stern-Gerlach apparatus (aligned with the z-axis), how many different paths (trajectories) can the atom take on its way out of that apparatus?

Answer: Since ℓ=1 implies three different possible values for m (+1, 0 ,-1), there are three possible trajectories (corresponding to these three magnetic quantum numbers) that the outgoing atoms can take.

d) For the case above, if we observe that the atom takes the top-most of all possible trajectories (meaning its magnetic moment must be aligned in the negative z-direction), by what fraction of its binding energy does its energy in a 5 Tesla magnetic field (pointing in positive z-direction) change from that of the same atom in a field-free region?

Note: The Rydberg constant is Ry = 13.6 eV and the electron Bohr magneton is

$\mu_B = \dfrac{e\hbar}{2m_e c} = 0.6\cdot 10^{-4}$ eV/Tesla , where $\vec{\mu}_\ell = \gamma_\ell \vec{\mathbf{L}} = -\mu_B \dfrac{\vec{\mathbf{L}}}{\hbar}$

Answer: The top-most trajectory corresponds to m = 1 (because of the negative value of the electron charge, this follows from the magnetic moment pointing down). Plugging in the numbers in the expression for the Hamiltonian, $H = -\gamma\mathbf{L}_z B$, yields $3\cdot 10^{-4}$ eV. This is about 0.01% of the binding energy, $Ry/n^2 = 3.4$ eV.

Problem 5) – *Extra Credit*

a) According to Heisenberg's uncertainty relationship, $\Delta\mathbf{O}_\psi \Delta\mathbf{Q}_\psi \geq \frac{1}{2}\left|\langle\psi|[\mathbf{O},\mathbf{Q}]|\psi\rangle\right|$, what must be the minimum uncertainty product $\Delta\mathbf{L}_x\Delta\mathbf{L}_y$ for measurements of the angular momentum $\mathbf{L}_x$,$\mathbf{L}_y$ if the wave function is in a (normalized) eigenstate $|\ell,m\rangle$ to $\mathbf{L}^2$ and $\mathbf{L}_z$ with eigenvalues $\ell(\ell+1)\hbar^2$, $m\hbar$? On the other hand, given that $\langle\mathbf{L}_x\rangle = \langle\mathbf{L}_y\rangle = 0$ in this state and that $\mathbf{L}^2 = \mathbf{L}_x^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2$, what is the **actual** value of this product $\Delta\mathbf{L}_x\Delta\mathbf{L}_y$ (assuming $\Delta\mathbf{L}_x = \Delta\mathbf{L}_y$)? Does Heisenbergs relationship hold in this case? Under what circumstances (what values of l, m) is it fulfilled exactly?

Answer: We know that $\left[\mathbf{L}_x,\mathbf{L}_y\right] = i\hbar\mathbf{L}_z$ and therefore

$\Delta\mathbf{L}_x\Delta\mathbf{L}_y \geq \frac{1}{2}\left|\left\langle \ell,m\middle|\left[\mathbf{L}_x,\mathbf{L}_y\right]\middle|\ell,m\right\rangle\right| = \frac{1}{2}\left|\left\langle \ell,m\middle|i\hbar\mathbf{L}_z\middle|\ell,m\right\rangle\right| = \frac{\hbar^2|m|}{2}$ according to Heisenberg.

On the other hand, we know that $\left(\Delta\mathbf{L}_x\right)^2 = \left\langle\mathbf{L}_x^2\right\rangle - \left\langle\mathbf{L}_x\right\rangle^2 = \left\langle\mathbf{L}_x^2\right\rangle$ (and similar for $\Delta\mathbf{L}_y$) so we can write

$$\hbar^2\ell(\ell+1) = \left\langle\mathbf{L}^2\right\rangle = \left\langle\mathbf{L}_x^2\right\rangle + \left\langle\mathbf{L}_y^2\right\rangle + \left\langle\mathbf{L}_z^2\right\rangle = 2\left(\Delta\mathbf{L}_x\right)^2 + \left\langle\ell,m\middle|\mathbf{L}_z^2\middle|\ell,m\right\rangle =$$

$$= 2\left(\Delta\mathbf{L}_x\right)^2 + \left|\mathbf{L}_z\left|\ell,m\right\rangle\right|^2 = 2\left(\Delta\mathbf{L}_x\right)^2 + \left(m\hbar\right)^2 \text{ } (\mathbf{L}_z \text{ is hermitian}) \Rightarrow$$

$$\Delta\mathbf{L}_x = \Delta\mathbf{L}_y = \sqrt{\hbar^2\frac{\ell(\ell+1)-m^2}{2}} \text{ and } \Delta\mathbf{L}_x\Delta\mathbf{L}_y = \hbar^2\frac{\ell(\ell+1)-m^2}{2}$$

which is clearly larger than the required by Heisenberg (m^2 is bounded by ℓ^2). The lower limit and the exact answer are only equal if $m = \ell$.

b) A particle of mass μ is bound in a spherically symmetric potential $V(r) = \begin{cases} -\frac{\hbar^2}{\mu r^2}, r \leq a \\ \infty, r > a \end{cases}$. What are the solutions to the Schrödinger Equation (= eigenstates to the Hamiltonian) for the total orbital angular momentum ℓ=1?

(Don't worry about absolute normalizations)

Answer: The Schrödinger equation for this situation reads as follows:

$$\frac{d^2U_{E1}}{dr^2} + \left[k^2 - \frac{-2\mu\hbar^2}{\hbar^2\mu r^2} - \frac{1(1+1)}{r^2}\right]U_{E1} = \frac{d^2U_{E1}}{dr^2} + k^2U_{E1} = 0 \text{ with } k = \frac{\sqrt{2\mu E}}{\hbar} \text{ for } r \leq a \text{ and } U_{E1} = 0$$

for $r > a$. The solution (with the requirement that $U_{E0} \to 0$ as $r \to 0$) reads

$$U_{E0}(r) = \begin{cases} A\sin kr, r \leq a \\ 0, r > a \end{cases}$$

The wave function must be continuous at $r = a$, which yields $A\sin ka = 0$ and therefore $k = \frac{n\pi}{a}$ or $E_n = \frac{n^2\pi^2\hbar^2}{2\mu a^2}$

Table of Clebsch-Gordan coefficients:

The nomenclature is $\left\langle j_1,m_1,j_2,m_2\middle|J,M\right\rangle$:

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\middle| 1,1\right\rangle = 1$$

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\middle| 1,0\right\rangle = \left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\middle| 1,0\right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\middle| 0,0\right\rangle = -\left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\middle| 0,0\right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\middle| 1,-1\right\rangle = 1$$

$$\left\langle 1,1,\frac{1}{2},\frac{1}{2}\middle| \frac{3}{2},\frac{3}{2}\right\rangle = 1$$

$$\left\langle 1,1,\frac{1}{2},-\frac{1}{2}\middle| \frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\ ; \quad \left\langle 1,0,\frac{1}{2},\frac{1}{2}\middle| \frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}$$

$$\left\langle 1,1,\frac{1}{2},-\frac{1}{2}\middle| \frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ ; \quad \left\langle 1,0,\frac{1}{2},\frac{1}{2}\middle| \frac{1}{2},\frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}$$

$$\left\langle 1,0,\frac{1}{2},-\frac{1}{2}\middle| \frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ ; \quad \left\langle 1,-1,\frac{1}{2},\frac{1}{2}\middle| \frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}$$

$$\left\langle 1,0,\frac{1}{2},-\frac{1}{2}\middle| \frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\ ; \quad \left\langle 1,-1,\frac{1}{2},\frac{1}{2}\middle| \frac{1}{2},-\frac{1}{2}\right\rangle = -\sqrt{\frac{2}{3}}$$

$$\left\langle 1,-1,\frac{1}{2},-\frac{1}{2}\middle| \frac{3}{2},-\frac{3}{2}\right\rangle = 1$$

(all others not listed are zero)