1-D Translations

Consider the operator

$$\mathbf{T}(\Delta x)\left|x\right\rangle = \left|x+\Delta x\right\rangle$$

Obviously this operator represents a translation in the x direction by some distance Δx.
For an infinitesimal shift, $\epsilon \to 0$, we would have $\mathbf{T}(\epsilon)\left|x\right\rangle = \left|x+\epsilon\right\rangle$. Applying this translation operator to an arbitrary state vector, $\left|\psi\right\rangle$ yields

$$\mathbf{T}(\epsilon)\left|\psi\right\rangle = \left|\psi'\right\rangle$$

In order for this operator to be useful, the following properties must be true:

- If $\left|\psi\right|^2 = 1$, then $\left|\psi'\right|^2 = 1$
- $\mathbf{T}(\Delta x \to 0) \to \mathbb{1}$
- $\mathbf{T}(\Delta x_1)\mathbf{T}(\Delta x_2) = \mathbf{T}(\Delta x_1 + \Delta x_2)$

From the first requirement we have

$$\left\langle \psi' | \psi' \right\rangle = \left\langle \psi | \mathbf{T}^\dagger(\epsilon)\mathbf{T}(\epsilon) | \psi \right\rangle = 1$$

Since this must be valid for **ANY** arbitrary state vector, it must be the case that $\mathbf{T}$ is unitary, or $\mathbf{T}^\dagger(\epsilon)\mathbf{T}(\epsilon) = \mathbf{T}(\epsilon)\mathbf{T}^\dagger(\epsilon) = \mathbb{1}$.
Let's assume that $\mathbf{T}$ can be represented as a linear combination of the unit operator and some arbitrary operator $\mathbf{G}$ such that

$$\mathbf{T}(\epsilon) = \mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{G}$$

and

$$\mathbf{T}^\dagger(\epsilon) = \mathbb{1} + \frac{i\epsilon}{\hbar}\mathbf{G}^\dagger$$

To find what $\mathbf{G}$ is, let's calculate $\mathbf{T}^\dagger(\epsilon)\mathbf{T}(\epsilon)$. Dropping terms with order higher than ϵ (since it is infinitesimally small anyway), we see that

$$\begin{aligned}
\mathbf{T}^\dagger(\epsilon)\mathbf{T}(\epsilon) &= \left(\mathbb{1} + \frac{i\epsilon}{\hbar}\mathbf{G}^\dagger\right)\left(\mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{G}\right) \\
&= \mathbb{1} + \frac{i\epsilon}{\hbar}\mathbf{G}^\dagger - \frac{i\epsilon}{\hbar}\mathbf{G} \\
&= \mathbb{1} + \frac{i\epsilon}{\hbar}(\mathbf{G}^\dagger - \mathbf{G}) \\
&\therefore \mathbf{G} \text{ is Hermitian}
\end{aligned}$$

Now that we know $\mathbf{G}$ is Hermitian, let's examine the commutator between $\mathbf{T}(\epsilon)$ and the $\mathbf{X}$ operator:

$$\mathbf{X}\mathbf{T}(\epsilon)|x\rangle = \mathbf{X}|x+\epsilon\rangle = (x+\epsilon)|x+\epsilon\rangle$$

$$\mathbf{T}(\epsilon)\mathbf{X}|x\rangle = \mathbf{T}(\epsilon)x|x\rangle = x|x+\epsilon\rangle$$

So, a translation following by a measurement of the position yields a different result than first measuring the position followed by a translation (which should be no great shock).

$$[\mathbf{X}, \mathbf{T}(\epsilon)] = \epsilon\mathbf{T}(\epsilon)$$

$$\left[\mathbf{X}, \mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{G}\right] = \epsilon\left(\mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{G}\right)$$

Again, we drop terms with order higher than ϵ and note that the unit operator commutes with anything.

$$[\mathbf{X}, \mathbb{1}] - \frac{i\epsilon}{\hbar}[\mathbf{X}, \mathbf{G}] = \epsilon$$

$$\rightarrow [\mathbf{X}, \mathbf{G}] = i\hbar$$

$$\rightarrow \mathbf{G} = \mathbf{P}$$

Therefore, the generator for a translation is simply the momentum operator, and we have $\mathbf{T}(\epsilon) = \mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{P}$.

All of these derivation was used on the assumption that the size of the translation, ϵ, is infinitesimally small, but what if the desired shift is some finite distance Δx? In that case we break the translation up into N small translations, apply the translation N times, and allow N to go to infinity.

$$\mathbf{T}(\Delta x) = \lim_{N\to\infty}\left(\mathbf{T}\left(\frac{\Delta x}{N}\right)\right)^N = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\frac{\Delta x}{N}\mathbf{P}\right)^N = e^{\dfrac{-i\Delta x\mathbf{P}}{\hbar}}$$

2-D Rotations

We can derive the operator responsible for 2-D rotations in much the same way that we derived the 1-D translation operator. First let's note that, classically, a rotation through an angle φ_0 can be expressed using the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We define the operator $\mathbf{U}[R_z(\varphi_0)]$ (causes a rotation through an angle φ_0 around the z axis) where

$$\mathbf{U}[R_z(\varphi_0)]\,|\psi\rangle = |\psi_R\rangle$$

It would be very odd to have a rotation operator that didn't rotate a position vector in the same way as a classical system. So, we must require that

$$\mathbf{U}[R_z(\varphi_0)]\,|x, y\rangle = |x\cos\varphi_0 - y\sin\varphi_0, x\sin\varphi_0 + y\cos\varphi_0\rangle = |R\vec{r}\rangle$$

Using the same arguments as with the 1-D translation operator, we let $\mathbf{U}[R_z(\varphi_0)] = \mathbb{1} - \frac{i\varphi_0}{\hbar}\mathbf{G}$. Now consider an infinitesimal rotation ϵ:

$$\begin{aligned}
\mathbf{U}[R_z(\epsilon)]\,|x, y\rangle &= |x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon\rangle \\
&= |x - \epsilon y, y + \epsilon x\rangle \\
&= \mathbf{T}_x(-\epsilon y)\mathbf{T}_y(\epsilon x)\,|x, y\rangle \\
&= \left(\mathbb{1} - \frac{i(-\epsilon y)}{\hbar}\mathbf{P}_x\right)\left(\mathbb{1} - \frac{i(\epsilon x)}{\hbar}\mathbf{P}_\mathbf{y}\right)|x, y\rangle \\
&= \left(\mathbb{1} + \frac{i\epsilon y}{\hbar}\mathbf{P}_x - \frac{i\epsilon x}{\hbar}\mathbf{P}_\mathbf{y}\right)|x, y\rangle
\end{aligned}$$

Since $[R_i, P_j] = \delta_{i,j}$, both x and y can be "promoted" to operators. We also note that this relationship is true for any vector $|x, y\rangle$, which allows us to relate the operators themselves. So we have

$$\mathbf{U}[R_z(\epsilon)] = \mathbb{1} - \frac{i\epsilon}{\hbar}(\mathbf{X}\mathbf{P}_y - \mathbf{Y}\mathbf{P}_x) = \mathbb{1} - \frac{i\epsilon}{\hbar}\mathbf{L}_z$$

Rotation by a finite angle φ_0 can be obtained in a similar way to translating by a finite distance:

$$\mathbf{U}[R_z(\varphi_0)] = e^{\frac{-i\varphi_0 \mathbf{L}_z}{\hbar}}$$

A very convenient coordinate system to use when working with this operator is polar coordinates. In polar coordinates, a rotation will only cause a change in the ϕ coordinate.

$$\mathbf{U}[R_z(\varphi_0)] \left|\rho, \varphi\right\rangle_c = \left|\rho, \varphi + \varphi_0\right\rangle_c$$

Here, we introduce a new *labeling* for our basis vectors - note that they are still the same position eigenstates as before, just labeled with (ρ, φ) instead of (x, y). In fact, we simply define

$$\left|\rho, \varphi\right\rangle_c = \left|x = \rho\cos\varphi, y = \rho\sin\varphi\right\rangle .$$

We can then introduce for any ket $|\psi\rangle$ its representation in these new variables as

$$\psi_c(\rho, \varphi) := \langle \rho, \varphi|\psi\rangle = \psi(\rho\cos\varphi, \rho\sin\varphi) = \langle x = \rho\cos\varphi, y = \rho\sin\varphi|\psi\rangle .$$

Note that, by the laws of integration,

$$\int\int \rho d\rho d\varphi \psi_c^*(\rho, \varphi)\psi_c(\rho, \varphi) = \int\int dxdy\psi^*(x, y)\psi(x, y) = 1$$

for proper normalization. This implies

$$\int\int d\rho d\varphi \left|\rho, \varphi\right\rangle_c \rho \left\langle\rho, \varphi\right| = \mathbf{1}.$$

For reference, we note the normalization of the new way of writing our basis vectors:

$$\begin{aligned}\left\langle\rho', \varphi'|\rho, \varphi\right\rangle_c &= \langle \rho'\cos\varphi', \rho'\sin\varphi'|\rho\cos\varphi, \rho\sin\varphi\rangle \\ &= \delta(\rho'\cos\varphi' - \rho\cos\varphi)\delta(\rho'\sin\varphi' - \rho\sin\varphi).\end{aligned}$$

Using $\delta(f(x) - b) = \delta(x - f^{-1}(b))/|f'(x)|$, we can evaluate this expression as

$$\left\langle\rho', \varphi'|\rho, \varphi\right\rangle_c = \frac{1}{\cos\varphi'}\delta\left(\rho' - \rho\frac{\cos\varphi}{\cos\varphi'}\right)\delta(\rho\cos\varphi\tan\varphi' - \rho\sin\varphi)$$

$$= \frac{1}{\cos\varphi'}\delta\left(\rho' - \rho\frac{\cos\varphi}{\cos\varphi'}\right)\frac{\cos^2\varphi'}{\rho\cos\varphi}\delta(\varphi' - \arctan(\sin\varphi/\cos\varphi)) = \frac{1}{\rho}\delta(\rho' - \rho)\delta(\varphi' - \varphi).$$

To find a representation for $\mathbf{L}_z$ in polar coordinates, consider an arbitrary wave function that has been rotated by an infinitesimal amount in polar coordinates:

$$\psi_c(\rho, \varphi + \epsilon) = \langle \rho, \varphi + \epsilon | \psi \rangle$$

$$= \langle \rho, \varphi | \mathbf{U}[R_z(\epsilon)] | \psi \rangle$$

$$= \left\langle \rho, \varphi | \mathbb{1} - \frac{i\epsilon}{\hbar} \mathbf{L}_z | \psi \right\rangle$$

$$= \psi_c(\rho, \varphi) + \frac{i\epsilon}{\hbar} \langle \rho, \varphi | \mathbf{L}_z | \psi \rangle$$

We also note that

$$\psi_c(\rho, \varphi + \epsilon) = \psi_c(\rho, \varphi) + \epsilon \frac{\partial}{\partial \varphi} \psi_c(\rho, \varphi) + \mathcal{O}(\epsilon^2)$$

So,

$$\frac{i}{\hbar} \langle \rho, \varphi | \mathbf{L}_z | \psi \rangle = \frac{\partial}{\partial \varphi} \psi_c(\rho, \varphi)$$

$$\rightarrow \langle \rho, \varphi | \, \mathbf{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \langle \rho, \varphi |$$

Now that we have a representation for $\mathbf{L}_z$, it would be useful to know its related eigenvalues. If $|l_z\rangle$ is an eigenfunction of $\mathbf{L}_z$, then the related eigenvalue will be l_z. Using the derivative form of $\mathbf{L}_z$ will give

$$-i\hbar \frac{\partial}{\partial \varphi} \psi_{l_z}(\rho, \varphi) = l_z \psi_{l_z}(\rho, \varphi)$$

$$\rightarrow \psi_{l_z}(\rho, \varphi) = A R(\rho) e^{\frac{i l_z \varphi}{\hbar}}$$

To find l_z we note that $l_z/\hbar$ must be an integer (since we require $\psi(\rho, 2\pi) = \psi(\rho, 0)$). So, l_z is quantized. More specifically,

$$\frac{2\pi l_z}{\hbar} = 2\pi n$$

$$\rightarrow l_z = \hbar n$$