Graduate Quantum Mechanics – Midterm Exam - Solution

Instruction: Pick only **one** of the first two problems (1 **or** 2) and solve it during the allotted time. You may take the other one home and turn it in for extra credit (at our next regular class, Tuesday 10/30).

Problem 1)

Let |0> be the ground state of the Harmonic Oscillator in one dimension (frequency ω, mass m), with energy eigenvalue $E = \frac{1}{2}\hbar\omega$. Let's define the "annihilation operator" as

$\mathbf{a} = \sqrt{\dfrac{m\omega}{2\hbar}}\mathbf{X} + i\sqrt{\dfrac{1}{2m\hbar\omega}}\mathbf{P}$ (with the usual definitions of the position and momentum operators). We

know that the ground state must fulfill the equation $\mathbf{a}|0> = 0$. Write this equation in the position basis and solve it to find the ground state wave function for the Harmonic Oscillator.

Note: $\langle x|0\rangle = \varphi_0(x); \langle x|X = x\langle x|; \langle x|P = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x}\langle x|$.

Ans.:

$$0 = \langle x|a|0\rangle = \sqrt{\dfrac{m\omega}{2\hbar}}x\varphi_0(x) + i\sqrt{\dfrac{1}{2m\hbar\omega}}\dfrac{\hbar}{i}\dfrac{\partial}{\partial x}\varphi_0(x) = \sqrt{\dfrac{m\omega}{2\hbar}}\left(x\varphi_0(x) + \dfrac{\hbar}{m\omega}\varphi_0'(x)\right) \Rightarrow$$

$$\dfrac{d\varphi_0}{\varphi_0} = d\ln\varphi_0 = -\dfrac{m\omega}{\hbar}x dx = -\dfrac{m\omega}{2\hbar}dx^2 \Rightarrow \varphi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

Problem 2)

Consider the following simplified model of neutrino oscillation:
Assume that there are two different neutrino types represented by state vectors $|v_1\rangle$ and $|v_2\rangle$. These are the eigenstates of the Hamiltonian in the 2-dimensional Hilbert Space spanned by these two basis vectors:
$$\mathbf{H}|v_1\rangle = E_1|v_1\rangle; \mathbf{H}|v_2\rangle = E_2|v_2\rangle$$
However, when neutrinos are produced in a weak interaction, they are always produced in so-called "flavor eigenstates", either as electron-neutrinos: $|v_e\rangle = \cos\theta|v_1\rangle + \sin\theta|v_2\rangle$

or as muon-neutrinos: $|v_\mu\rangle = -\sin\theta|v_1\rangle + \cos\theta|v_2\rangle$

Assume a neutrino is produced at time $t = 0$ as a pure electron neutrino. Calculate the probability that, after some time $t = T$, it has converted to a muon neutrino (i.e., a measurement of the neutrino flavor yields the answer "muon type").

Ans.:

We begin by writing to down the time evolution for the electron neutrino using the eigenstates of the Hamiltonian:
$$|\psi(t)\rangle = \sum\langle v_i|\psi(0)\rangle e^{-iE_i t/\hbar}|v_i\rangle = e^{-iE_1 t/\hbar}\cos\theta|v_1\rangle + e^{-iE_2 t/\hbar}\sin\theta|v_2\rangle$$

The probability to find the eigenstate of a muon neutrino is given by

$$P(\nu_\mu) = \left| \langle \nu_\mu | \psi(T) \rangle \right|^2 = \left| -e^{-iE_1 T/\hbar} \cos\theta \sin\theta + e^{-iE_2 T/\hbar} \cos\theta \sin\theta \right|^2 = \sin^2 2\theta \frac{1}{4} \left(e^{-iE_2 T/\hbar} - e^{-iE_1 T/\hbar} \right)\left(e^{+iE_2 T/\hbar} - e^{+iE_1 T/\hbar} \right)$$

$$= \sin^2 2\theta \frac{1}{4}\left(1+1 - e^{+i(E_1-E_2)T/\hbar} - e^{-i(E_1-E_2)T/\hbar}\right) = \frac{1}{2}\sin^2 2\theta\left(1 - \cos\frac{E_1-E_2}{\hbar}T\right) = \sin^2 2\theta \sin^2\frac{(E_1-E_2)T}{2\hbar}$$

Problem 3)

Show that for a quantum particle moving freely in one dimension (H = P²/2m) the uncertainty in its momentum, Δp, is constant (doesn't change with time).
Hint: You can use commutators to solve this quickly!

Ans.:

From our postulates, we know that the time dependence of the expectation value for any operator is given by $\frac{d}{dt}\langle\Omega\rangle = \frac{1}{i\hbar}\langle[\Omega, H]\rangle$. For the free Hamiltonian, both P and P² commute with it. Thus, their expectations values are unchanged with time, and so is $\Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2}$.

Problem 4)

In which ways does a "Gaussian wave packet" governed by a free Hamiltonian in one dimension be-have just like a classical free particle? What is the most striking difference between the classical and the correct quantum mechanical behavior in this case?

Ans.:

The Gaussian wave packet is the best possible approximation of a localized particle with well-defined momentum. In particular, its centroid moves with constant velocity $v = p/m$, just like the position of a classical free particle with the same momentum. On the other hand, there is a lower limit to how well we can define both position and momentum (the product of their uncertainties is $\hbar/2$), and the uncertainty in position even grows over time ("spreading of the wave packet"), which clearly is not classical behavior.

Problem 5)

Assume you have two observables (Ω, Λ) that are compatible (i.e., they have a common set of eigen-states). Further assume that each eigenvalue (out of several) of either observable is degenerate, meaning for each eigenvalue ω_i of Ω there are several eigenstates and similarly for Λ. However, also assume that each **pair** of eigenvalues (ω_i, λ_j) uniquely defines one and only one state of the system (except for mul-tiplication with a complex number), so that the set of eigenvectors $|\omega_i, \lambda_j\rangle$ is a complete and orthonor-mal basis of the Hilbert Space.
Initially, the system is in some state $|\psi\rangle$ that is completely unknown to me. I first measure observable Ω, with result ω_1, immediately followed by a measurement of Λ, with result λ_1, and then again by a

measurement of Ω.

    1.) Describe the extent of our knowledge about which state the system is in after each of the three measurements. Be as precise as possible: What do we and don't we know about the state at each point?

    2.) At which point along this chain do I know everything there is to know about the (present) state of the system?

    3.) For which of the three measurements can I predict the exact outcome?

    4.) For which of the same three measurements can I predict the **probability** of a given possible outcome?

    Now assume that I **do** completely know the state $|\psi>$ of the system initially (before the first measurement), but it is not an eigenstate of either Ω or Λ. Answer the same 4 questions again in light of this additional information:

    5.) Describe the extent of our knowledge about which state the system is in after each of the three measurements. Be as precise as possible: What do we and don't we know about the state at each point?

    6.) At which point along this chain do I know everything there is to know about the (present) state of the system?

    7.) For which of the three measurements can I predict the exact outcome?

    8.) For which of the same three measurements can I predict the **probability** of a given possible outcome?

Ans.:

    1.) After the first measurement, I only know that the state is now within the subspace of the Hilbert Space that is spanned by the eigenvectors of Ω with eigenvalue ω_1, but any linear combination of them is possible.

Since the 2 observables commute, measuring Λ will not move the state out of this subspace. So now we have a complete set of eigenvalues specified for both Ω and Λ, meaning we know the exact state (up to a single multiplicative factor) the system is in.

The final measurement of Ω doesn't change anything.

2.) As outlined above, this happens after the 2nd measurement.

3.) Accordingly, I can predict the outcome of the final measurement with certainty: ω_1.

4.) Since I don't know the state before any of the first two measurements completely, I cannot even predict the probability for each possible outcome for these two (except that the outcome must be an eigenvalue of the operator measured). For the last measurement I can of course give the probability for the outcome ω_1: it is 100%.

5.) In the second case, I know the state of the system at every stage along the sequence: initially, and after each measurement. This is because after I know the outcome of a measurement, I only have to apply the projection operator belonging to the respective eigenvalue to the state before the measurement to get the (unnormalized but normalizable) state vector after the measurement.

6.) See 5.)

7.) In spite of my greater knowledge, I still can only predict the outcome of the last measurement with certainty, since for the first two measurements, the system is not in an eigenstate of the operator to be measured.

8.) Since I know the state vector at each point in time, I can predict the probability for any outcome of any of the measurements.