Chapter 2

Motion along a straight line

Kinematics & Dynamics

Kinematics:
Description of Motion without regard to its cause.

Dynamics:
Study of principles that relate motion to its cause.

Basic physical variables in kinematics and dynamics:

- Time
- Position
- Displacement
- Velocity
- Acceleration
Part 1

Time, Position, Velocity, Acceleration

Time

Time: that part of existence which is measured in seconds, minutes, hours, days, weeks, months, years, etc., or this process considered as a whole

Cambridge Dictionary

Time: something that is measured in minutes, hours, years etc using clocks

Clocks: an instrument in a room or in a public building that shows what time it is

Longman Dictionary of Contemporary English
**Time**

"Time is like a river flowing." – Newton

"Zeit ist was enie Uhr mass." – Einstein

("Time is what a clock measures.")

"Time is space between events" – Feynman

"Time is that quality of nature which keeps events from happening all at once" – Anonymous

Time is neither young nor old

a beginning or an end,

now or forever;

time is the empty distance in between.

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**Position and Displacement**

- **Position vector** defines location of point P in space:
  - In three dimensions: \[ \vec{r} \]
  - In two dimensions: \[ \vec{r} \]

- **Displacement** is change in position: \[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \]

Note:

- **graphical** and **algebraic** methods of vector addition agree,

\[ \vec{r}_1 + \Delta \vec{r} = \vec{r}_1 + \vec{r}_2 - \vec{r}_1 = \vec{r}_2 \]
Trajectory

- **Trajectory** is formed by successive displacements, in the limit when they become infinitesimally small.

- As $P_2 \to P_1$, the limiting direction of displacement (from $P_1$ to $P_2$ on the curve) is along the tangent to the curve at $P_1$.

- **Infinitesimally small** $\Delta \vec{r}$ is denoted by $d\vec{r}$:

$$d\vec{r} = \lim_{P_2 \to P_1} \Delta \vec{r}$$

and is called instantaneous displacement.

Velocity

Velocity is the rate at which displacement changes over an interval of time:

Average velocity:

$$\langle \vec{v} \rangle \equiv \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

**Note:** $\langle \vec{v} \rangle$, as a product of a vector $\Delta \vec{r}$ and a scalar $1/\Delta t$, is in the same direction as $\Delta \vec{r}$. 
Velocity and Speed

Note distinction between average velocity of a particle and its average speed as it travels from point $P_1$ to point $P_2$.

Average speed:

$$\langle s \rangle \equiv \frac{\Delta l}{\Delta t}$$

Notice:

$\langle s \rangle$ is a scalar, while $\langle \vec{v} \rangle$ is a vector.

Furthermore: $\langle s \rangle$ is always greater than $|\langle \vec{v} \rangle|$ unless the trajectory is a straight line.

Instantaneous velocity

Definition

$$\vec{v} = \lim_{\Delta t \to 0} \langle \vec{v} \rangle = \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Notice:

$\vec{v}$ is tangent to trajectory.

Instantaneous velocity $\vec{v}$ is velocity at a point on the trajectory at a particular instant of time $t$.

Instantaneous speed $s$ is just the magnitude of the instantaneous velocity $\vec{v}$: $s = |\vec{v}|$.
**Acceleration**

Acceleration is the rate at which velocity changes over an interval of time.

\[
\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}
\]

**Instantaneous acceleration:**

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}
\]

Note: there is change in velocity even if \( \vec{v}_2 \) and \( \vec{v}_1 \) have the same magnitudes at both \( P_1 \) and \( P_2 \).

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**Analytic representation**

Instantaneous position, displacement, velocity, acceleration can be expressed in terms of coordinates and unit vectors (we will choose Cartesian coordinates in two dimensions).

**Position:**

\[
\vec{r} = \hat{i} x + \hat{j} y
\]

**Displacement:**

\[
d\vec{r} = d (\hat{i} x + \hat{j} y) = \hat{i} dx + \hat{j} dy
\]
Velocity and acceleration

Velocity:
\[
\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}
\]

Obviously,
\[
v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt}
\]

Acceleration:
\[
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} \right) = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j}
\]

Obviously,
\[
a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}
\]

Generalization for three dimensions should be clear.

One-Dimensional (1-D) Motion

- **Equations** for 1-D motion along x-axis:

  \[
\begin{align*}
  \vec{r} &= x \hat{i} \\
  \vec{v} &= \frac{dx}{dt} \hat{i} = \frac{d}{dt} (x \hat{i}) \\
  \vec{a} &= \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \hat{i} = \frac{dv_x}{dt} \hat{i} = \frac{d^2x}{dt^2} \hat{i} = a_x \hat{i}
\end{align*}
\]

We will suppress factor \( \hat{i} \) for one-dimensional motion

But: \( \hat{i} \) is still there:

\[d\vec{r} = dx \hat{i}, \quad \text{with} \ dx > 0 \ \text{for displacement in} \ +\hat{i} \ \text{direction;}
\]

and \( dx < 0 \) for \(-\hat{i}\) direction

- Knowing position \( x(t) \) we can calculate velocity and acceleration by differentiating \( x(t) \) with respect to \( t \)

- Knowing acceleration, velocity and position is obtained by integration, or anti-differentiation over \( t \)
Part 2

Motion with constant Acceleration

Constant Acceleration

- Easiest case: constant acceleration
  \[ a_x = a_0, \quad a_0 = \text{constant} \]
  \[ \Rightarrow \quad \frac{dv_x}{dt} = a_0 \quad \text{or} \]
  \[ dv_x = a_0 dt \]

- Integrate to get the velocity:
  \[ \int dv_x = \int a_0 \, dt \quad \Rightarrow \quad v_x = a_0 t + C_1 \]
  with \( C_1 = \text{constant} \)

Use initial condition: \( v_x = v_{x0} \) when \( t = 0 \):
\[ v_x(t) = a_0 t + v_{x0} \]
Calculating position

- Equation for position:

\[
\frac{dx}{dt} = v_x = a_0 t + v_{x_0} \Rightarrow dx = v_x \, dt = (a_0 t + v_{x_0}) \, dt
\]

- Integrate once more to get the position:

\[
\int dx = \int (a_0 t + v_{x_0}) \, dt
\]

\[\Rightarrow x(t) = \frac{1}{2} a_0 t^2 + v_{x_0} t + C_2\]

with \(C_2 = \text{constant}\)

Use initial condition: \(x = x_0\) when \(t = 0\):

\[\Rightarrow x(t) = \frac{1}{2} a_0 t^2 + v_{x_0} t + x_0\]

Summary table of results

- Summarizing results for 1-D motion along x-axis:

\[
\begin{align*}
a_x &= a_0 \\
v_x &= a_0 t + v_{x_0} \\
x &= \frac{1}{2} a_0 t^2 + v_{x_0} t + x_0
\end{align*}
\]

Any motion in 1-D with constant acceleration is described by this set of equations: different physical situations differ only in their initial conditions
1D motion with constant acceleration $a$

\[
\begin{align*}
\dot{v} &= v_0 + at \\
\dot{x} &= x_0 + v_0 t + \frac{1}{2} at^2
\end{align*}
\]

the system of equations with six variables

max: two unknowns

case 1: we may eliminate time from the system

\[v^2 = v_0^2 + 2a(x - x_0)\]

case 2: we may eliminate acceleration from the system

\[x = x_0 + \frac{1}{2} (v_0 + v)t\]

---

**example**

Car travels at 90 km/h (56 miles/h).

Driver applies brakes (assume constant acceleration) and brings car to a stop in 250 m (820 ft).

What is the value of acceleration?

Initial conditions:

\[x_0 = 0 \quad , \quad v_{x0} = 90 \text{ km/h}\]

Final conditions at moment $T$ when car stopped:

\[x(T) = X = 250 \text{ m} \quad , \quad v_x(T) = 0\]
example

Since \( x = X \) when \( t = T \), we have

\[
X = \frac{1}{2}a_0 T^2 + v_x 0 + x_0
\]

(1)

We also have \( v_x = 0 \) when \( t = T \).

This gives a relation between two unknowns, \( a_0 \) and \( T \):

\[
0 = v_x = a_0 T + v_x 0 \Rightarrow T = -\frac{v_x 0}{a_0}
\]

(2)

Substituting \( T = -\frac{v_x 0}{a_0} \) into Eq. (1):

\[
X = \frac{a_0}{2} \left( -\frac{v_x 0}{a_0} \right)^2 + v_x 0 \left( -\frac{v_x 0}{a_0} \right) + x_0
\]

\[
= \frac{a_0}{2} \left( \frac{v_x^2 0}{a_0^2} \right) - \frac{v_x^2 0}{a_0} + x_0 = -\frac{v_x^2 0}{2a_0} + x_0
\]

example

Relation between stopping distance \( X - x_0 \), initial velocity \( v_x 0 \), and acceleration \( a_0 \)

\[
X - x_0 = -\frac{v_x^2 0}{2a_0}
\]

Solution:

\[
a_0 = -\frac{v_x^2 0}{2(X - x_0)}
\]

Numerically, using

\[
90 \text{ km/h} = (9 \cdot 10^4 \text{ m})/(3.6 \cdot 10^3 \text{ s}) = (25 \text{ m/s})^2;
\]

\[
a_0 = -\frac{(90 \text{ km/h})^2}{2 (250 - 0) \text{ m}} = \frac{(25 \text{ m/s})^2}{2 (250 \text{ m})} = -1.25 \text{ m/s}^2
\]
The catapult of the aircraft carrier USS Abraham Lincoln accelerates an F/A-18 Hornet jet fighter from rest to a takeoff speed of 173 mph in a distance of 307 ft. Assume constant acceleration.

1. Calculate the acceleration of the fighter in m/s.

2. Calculate the time required for the fighter to accelerate to takeoff speed.

Given:

\[ v_0 = 0.0 \text{ m/s} \]

\[ x_0 = 0.0 \text{ m} \]

\[ x = 307 \text{ ft} = 93.57 \text{ m} \]

\[ v = 173.0 \text{ mph} = 77.33 \text{ m/s} \]

Calculations:

\[ a = \frac{v^2 - v_0^2}{2(x - x_0)} = 32.0 \text{ m/s}^2 \]

problem 2.36

At the instant the traffic light turns green, a car that has been waiting at an intersection starts ahead with a constant acceleration of 3.2 m/s². At the same instant a truck, traveling with a constant speed of 20.0 m/s, overtakes and passes the car.

How far beyond its starting point does the car overtake the truck?

Given:

\[ x_0 = 0.0 \text{ m} \]

\[ v_{0t} = 20.0 \text{ m/s} \]

\[ v_{0c} = 0.0 \text{ m/s} \]

\[ x_c = x_t \]

\[ a_t = 0.0 \text{ m/s}^2 \]

\[ a_c = 3.2 \text{ m/s}^2 \]

2 objects = 4 equations

time is the same

\[ x_t = v_{0t}t + x_c = 0.5a_c t^2 \]

\[ t = \frac{2v_{0t}}{a_c} \]

\[ x_t = \frac{2v_{0t}^2}{a_c} \]
Part 3

Free Fall

Free fall

Another case of 1-D motion with constant acceleration – the constant acceleration of gravity

Table of equations:

\[
\begin{align*}
  a_y &= -g \\
  v_y &= -gt + v_{0y} \\
  y &= -\frac{1}{2}gt^2 + v_{0y}t + y_0
\end{align*}
\]
**example**

A person standing on a building 30.0 m high throws a ball upward with a velocity of 10.0 m/s.

- **How far does the ball rise before it starts to fall toward the ground?**

  **Initial conditions:**
  \[ y_0 = 30.0 \text{ m} \]
  \[ v_{y0} = 10.0 \text{ m/s} \]

  **Condition for \( t = T_1 \) when the ball stops rising:**
  \[ v_{y}(T_1) = 0 \]

  **Finding time \( T_1 \) during which the ball was rising:**
  At the moment \( t = T_1 \), the velocity-equation gives
  \[ 0 = -gt + v_{y0} \Rightarrow T_1 = \frac{v_{y0}}{g} = \frac{10.0 \text{ m/s}}{9.80 \text{ m/s}^2} = 1.02 \text{ s} \]

**Solution (cont)**

Finding how far the ball was rising:

Position-equation tells us now that

\[
y(T_1) = -\frac{1}{2}g \left( \frac{v_{y0}}{g} \right)^2 + v_{y0} \left( \frac{v_{y0}}{g} \right) + y_0
\]

\[
= \frac{v_{y0}^2}{2g} + y_0 = \frac{(10.0 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} + 30.0 \text{ m} = 35.1 \text{ m}
\]

**Corollary**

*If a body is thrown upward with velocity \( v_{y0} \), it rises to the height \( h = \frac{1}{2}v_{y0}^2/g \)*

\[ v^2 = v_0^2 + 2a(x - x_0) \]
The second question

- How long does the ball take (from the moment of release) to get to the ground?

Condition for $t = T_2$ when the ball strikes the ground: $y(T_2) = 0$

Finding time $T_2$ needed for the ball to reach the ground:
Position-equation at time $t = T_2$ tells us that
$$y(T_2) = 0 = -rac{1}{2}gt^2 + v_0T_2 + y_0$$

Need to solve quadratic equation
$$T_2^2 - 2(v_0/g)T_2 - 2y_0/g = 0$$

Solution for the second question

Rewriting quadratic equation
$$(T_2 - v_0/g)^2 = 2y_0/g + (v_0/g)^2$$

We can easily solve it:
$$T_2 = \frac{v_0}{g} \pm \sqrt{\left(\frac{v_0}{g}\right)^2 + 2\left(\frac{y_0}{g}\right)}$$
$$= \left(1.02 \pm \sqrt{(1.02)^2 + 2 \cdot 30.8}\right) \text{s}$$
$$= 1.02 \text{s} \pm 3.64 \text{s}$$

- We obtained two solutions $T_2^{(1)} = 4.66 \text{s}$ and $T_2^{(2)} = -2.62 \text{s}$

Which is correct?
Discussion of result for question 2

Of two solutions $T_2^{(1)} = 4.66 \, \text{s}$ and $T_2^{(2)} = -2.62 \, \text{s}$ only the first solution satisfies the boundary condition that the ball hits the ground after it was thrown from the building. The second solution satisfies another boundary condition: that the ball was on the ground level before it reached the top of the building. This could happen if the ball was thrown up from the ground with an appropriate velocity.

Our equations only “know” that the ball had upward velocity 10 m/s at the height 30 m. They do not “know” if the free-fall motion preceded this event, followed it, or both.

The third question

- What is the speed of the ball at the instant when it strikes the ground?
- Using $v_y(T_2) = v_{y0} - gT_2$ and

\[
T_2^{(1)} = \frac{v_{y0}}{g} + \sqrt{(\frac{v_{y0}}{g})^2 + 2(\frac{y_0}{g})}
\]

gives $v_y(T_2^{(1)}) = -\sqrt{\frac{v_{y0}^2}{g} + 2y_0g} = -26.2 \, \text{m/s}$

- Similarly, for $T_2^{(2)} = \frac{v_{y0}}{g} - \sqrt{(\frac{v_{y0}}{g})^2 + 2(\frac{y_0}{g})}$ we obtain $v_y(T_2^{(2)}) = \sqrt{\frac{v_{y0}^2}{g} + 2y_0g} = +26.2 \, \text{m/s}$

To have the velocity $+10 \, \text{m/s}$ at the height of 30 m, the ball should be thrown up from the ground with velocity $+26.2 \, \text{m/s}$ having the same magnitude that it will have when it will return to the ground.
Being practical: 1D motion with const. \(a\)

1. initial value problem:
   knowing initial conditions \((x_0, v_0)\)
   and acceleration \(a\), one may find
   \((x, v)\) at any moment in time.

\[
\begin{align*}
v & = v_0 + at \\
x & = x_0 + v_0t + \frac{1}{2}at^2
\end{align*}
\]

2. “who cares about time”
   then the system of equation can be reduced to a simple equation

\[
v^2 = v_0^2 + 2a(x - x_0)
\]

3. two object problem
   in this case we need to consider a system of 4 equations with the same time variables.
   In most problems we need only a couple equations from the system

\[
\begin{align*}
v_1 & = v_{10} + a_1t \\
x_1 & = x_{10} + v_{10}t + \frac{1}{2}a_1t^2 \\
v_2 & = v_{20} + a_2t \\
x_2 & = x_{20} + v_{20}t + \frac{1}{2}a_2t^2
\end{align*}
\]

different examples

**example**

The engineer of a passenger train traveling at 25.0 m/s sights a freight train whose caboose is 200 m ahead on the same track. The freight train is traveling at 15.0 m/s in the same direction as the passenger train.

The engineer of the passenger train immediately applies the brakes, causing a constant acceleration of -0.10 m/s\(^2\), while the freight train continues with constant speed.

1. Will the cows nearby witness a collision?
2. If so, where will it take place?

\[
\begin{align*}
x_{01} & = 0.0 \text{ m} & x_{02} & = 200.0 \text{ m} \\
v_{01} & = 25.0 \text{ m/s} & v_{02} & = 15.0 \text{ m/s} \\
a_1 & = -0.1 \text{ m/s}^2 & a_2 & = 0.0 \text{ m/s}^2
\end{align*}
\]
\[
\begin{align*}
v_1 &= v_{01} + a_1 t \\
x_1 &= x_{01} + v_{01}t + \frac{1}{2} a_1 t^2 \\
v_2 &= v_{02} + a_2 t \\
x_2 &= x_{02} + v_{02}t + \frac{1}{2} a_2 t^2
\end{align*}
\]

\[
\begin{align*}
v_1 &= v_{01} + a_1 t \\
x_1 &= 0.0 + v_{01}t + \frac{1}{2} a_1 t^2 \\
v_2 &= v_{02} \\
x_2 &= x_{02} + v_{02}t
\end{align*}
\]

\text{collision means } x_1 = x_2

\[
v_{01}t + \frac{1}{2} a_1 t^2 = x_{02} + v_{02}t
\]

\[
\begin{align*}
x_{01} &= 0.0 \text{ m} & x_{02} &= 200.0 \text{ m} \\
v_{01} &= 25.0 \text{ m/s} & v_{02} &= 15.0 \text{ m/s} \\
a_1 &= -0.1 \text{ m/s}^2 & a_2 &= 0.0 \text{ m/s}^2
\end{align*}
\]

solutions:
\[
\begin{align*}
t_1 &= 22.5 \text{ s} \\
t_2 &= 177.0 \text{ s}
\end{align*}
\]

negative solutions?