Chapter 3

Motion in Two or Three Dimensions

Outline

1. Position, velocity, acceleration
2. Motion in a plane (Set of equations)
3. Projectile Motion (Range, Height, Velocity, Trajectory)
4. Circular Motion (Polar coordinates, Time derivatives)
5. Relative Motion
Position, velocity, acceleration

1. Position

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

2. Average velocity

\[ \vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \]

3. Instantaneous velocity

\[ \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \]

\[ \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \]

\[ \vec{v} = \sqrt{v_x^2 + v_y^2 + v_z^2} \]

4. Average acceleration

\[ \vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} \]

5. Instantaneous acceleration

\[ \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \]

Part 1

Motion in a plane
Motion in Plane When Acceleration is Constant

- Consider a case when acceleration in a plane is constant $a = \vec{A} = \text{const}$
- In 2-D Cartesian coordinates, this can be written as $a_x \hat{i} + a_y \hat{j} = A_x \hat{i} + A_y \hat{j}$
- Or, in terms of velocity:
  \[
  \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} = A_x \hat{i} + A_y \hat{j}
  \]
- Since $\hat{i}$ and $\hat{j}$ are linearly independent, it is clear that
  \[
  \frac{dv_x}{dt} = A_x, \quad \frac{dv_y}{dt} = A_y
  \]

Motion that occurs in the $x$-direction is completely independent of motion that occurs in the $y$-direction.

Set of equations

- Each of these equations can be integrated to obtain components of velocity and position

\[
\begin{align*}
  a_x &= A_x \\
  v_x &= A_x t + v_{x0} \\
  x &= \frac{1}{2} A_x t^2 + v_{x0} t + x_0 \\
  v_y &= A_y t + v_{y0} \\
  y &= \frac{1}{2} A_y t^2 + v_{y0} t + y_0
\end{align*}
\]

Anything that can possibly be known about motion in the $xy$-plane at constant acceleration is contained within these equations.
Being practical: 2D motion with const. $a$

1. initial value problem:
   knowing initial conditions $(x_0, y_0, v_{x_0}, v_{y_0})$ and acceleration $(a_x, a_y)$ one may find position $(x, y)$ and velocity $(v_x, v_y)$ at any moment in time.

2. final value problem:
   when some (or all) or final values are known, one may find required initial conditions to satisfy the final values

3. a mixture of initial and final value problems (just algebra :-(

\[
\begin{align*}
  v_x &= v_{x_0} + a_x t \\
  x &= x_0 + v_{x_0} t + \frac{1}{2} a_x t^2 \\
  v_y &= v_{y_0} + a_y t \\
  y &= y_0 + v_{y_0} t + \frac{1}{2} a_y t^2
\end{align*}
\]

Part 2

Projectile Motion

- A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
- Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.

\[
\begin{align*}
  v_x &= v_{x_0} \\
  x &= x_0 + v_{x_0} t \\
  v_y &= v_{y_0} - gt \\
  y &= y_0 + v_{y_0} t - \frac{1}{2} gt^2
\end{align*}
\]
Projectile Motion – 2 D Example

- Projectile is launched at angle \( \theta_0 \)

Note that:
\[ v_{x0} = v_0 \cos \theta_0 \]
\[ v_{y0} = v_0 \sin \theta_0 \]

- Two sets of equations for motion in \( x \)-direction and \( y \)-direction can now be written as

\[
\begin{aligned}
\dot{v}_x &= \dot{A}_x = 0 \\
v_x &= v_{x0} = v_0 \cos \theta_0 \\
x &= x_0 t = v_0 t \cos \theta_0 \\
\dot{v}_y &= \dot{A}_y = -g \\
v_y &= -gt + v_{y0} \\
y &= -\frac{1}{2}gt^2 + v_{y0}t \sin \theta_0
\end{aligned}
\]

---

Projectile Motion – 2 D

At the top of the trajectory, the projectile has zero vertical velocity \( (v_y = 0) \), but its vertical acceleration is still \(-g\).

Vertically, the projectile exhibits constant-acceleration motion in response to the earth’s gravitational pull. Thus, its vertical velocity changes by equal amounts during equal time intervals.

Horizontally, the projectile exhibits constant-velocity motion. Its horizontal acceleration is zero, so it moves equal distances in equal time intervals.
Range of Projectile

Using equations for $y$ and $x$:

\[ 0 = -\frac{1}{2}gT_1^2 + v_0T_1 \sin \theta_0 , \quad R = v_0T_1 \cos \theta_0 \]

From 1st equation:
\[ T_1 = \frac{2v_0}{g} \sin \theta_0 \]

Substituting $T_1$ into 2nd equation:
\[ R = 2\frac{v_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{v_0^2}{g} \sin(2\theta_0) \]

Maximal range is reached when
\[ \sin(2\theta_0) = 1 \]
\[ \Rightarrow R_{\text{max}} = \frac{v_0^2}{g} \quad \text{for} \quad \theta_0 = 45^\circ \]

Maximum height reached by projectile

At maximum height:
\[ y(T_2) = H \]
\[ v_y(T_2) = 0 \]

Using equations for $v_y$ and $y$:
\[ 0 = -gT_2 + v_0 \sin \theta_0 , \quad H = -\frac{1}{2}gT_2^2 + v_0T_2 \sin \theta_0 \]

From 1st equation:
\[ T_2 = \frac{v_0}{g} \sin \theta_0 = \frac{1}{2} T_1 \]

Substituting $T_2$ into 2nd equation:
\[ H = \frac{v_0^2}{2g} \sin^2 \theta_0 = \frac{v_0^2}{2g} \]

Maximum height location $x(T_2) \equiv X$

from $x$-equation
\[ X = v_0T_2 \cos \theta_0 = \frac{v_0^2}{g} \cos \theta_0 \sin \theta_0 = \frac{R}{2} \]
Velocity at the ground

Impact occurs at time \( t = T \) \( \Rightarrow \)
velocity components at that time are
\[
 v_x = v_0 \cos \theta_0, \quad v_y = -gT_1 + v_0 \sin \theta_0 = -v_0 \sin \theta_0
\]

Speed of projectile at impact:
\[
v = \sqrt{v_x^2 + v_y^2} = \sqrt{(v_0 \cos \theta_0)^2 + (-v_0 \sin \theta_0)^2} = v_0
\]

Surprised?
This result will be obvious later when we learn about conservation of energy

Trajectory of the projectile

To find \( y \) as function of \( x \) extract \( t \) from equation for \( x(t) \):
\[
x = v_0 t \cos \theta_0 \Rightarrow t = \frac{x}{v_0 \cos \theta_0}
\]

and substitute \( t \) into equation for \( y \):
\[
y = -\frac{1}{2}gt^2 + v_0 t \sin \theta_0
\]
\[
= -\frac{1}{2}g \left( \frac{x}{v_0 \cos \theta_0} \right)^2 + v_0 \sin \theta_0 \left( \frac{x}{v_0 \cos \theta_0} \right)
\]
\[
= \left( -\frac{g}{2v_0^2 \cos^2 \theta_0} \right) x^2 + (\tan \theta_0) x
\]

Equation has the form \( y = -ax^2 + b \)
This is the equation of a parabola.
example 1: initial value problem


Using parabola equation

- The $x$-coordinate of the point where $y$ has the highest value (its maximum value) can be found using

\[
\left. \frac{dy}{dx} \right|_{x=X} = -\frac{g}{v_0^2 \cos^2 \theta_0} X + \tan \theta_0 = 0, \\
\Rightarrow X = \frac{v_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{R}{2}
\]
Problem

Figure shows a pirate ship 560 m from a fort defending the harbor entrance of an island. A defense cannon, located at sea level, fires balls at initial speed $v_0=82$ m/s.

(a) at what angle from the horizontal must a ball be fired to hit the ship?

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

two solutions:

$27^\circ$ and $63^\circ$.

Question

A battleship simultaneously fires two shells toward two enemy ships, one close by (A), one far away (B). The shells leave the battleship at different angles and travel along the parabolic trajectories indicated below. Which of the two enemy ships gets hit first? Do you need more information to answer the question?
Maximum height reached by projectile

At maximum height:

\[ y(T_2) = H \]
\[ v_y(T_2) = 0 \]

Using equations for \( v_y \) and \( y \):

\[ 0 = -gt_2 + v_0 \sin \theta_0 \]
\[ H = -\frac{1}{2}gt_2^2 + v_0t_2 \sin \theta_0 \]

From 1st equation:

\[ T_2 = \frac{v_0}{g} \sin \theta_0 = \frac{1}{2}T_1 \]

Substituting \( T_2 \) into 2nd equation:

\[ H = \frac{v_0^2}{2g} \sin^2 \theta_0 = \frac{v_0^2}{2g} \]

Maximum height location \( x(T_2) = X \)

From \( x \)-equation

\[ X = v_0T_2 \cos \theta_0 = \frac{v_0^2}{g} \cos \theta_0 \sin \theta_0 = \frac{R}{2} \]

---

**Question**

A cart on a roller-coaster rolls down the track shown below. As the cart rolls beyond the point shown, what happens to its speed and acceleration in the direction of motion?

1. Both decrease.
2. The speed decreases, but the acceleration increases.
3. Both remain constant.
4. The speed increases, but acceleration decreases.
5. Both increase.
6. Other
Question

At the same instant that you fire a bullet horizontally from a gun, you drop a bullet from the height of the barrel. If there is no air resistance, which bullet hits the ground first?

\[
\begin{align*}
  v_x &= v_{x0} \\
  x &= x_0 + v_{x0}t \\
  v_y &= v_{y0} - gt \\
  y &= y_0 + v_{y0}t - \frac{1}{2}gt^2
\end{align*}
\]

Question

When a rifle is fired at a distant target, the barrel is not lined up exactly on the target. Why not?

\[
\begin{align*}
  v_x &= v_{x0} \\
  x &= x_0 + v_{x0}t \\
  v_y &= v_{y0} - gt \\
  y &= y_0 + v_{y0}t - \frac{1}{2}gt^2
\end{align*}
\]
Problem

A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side the car is on is 21.3 m above the river, while the opposite side is a mere 1.8 m above the river. The river itself is a raging torrent 61.0 wide.

How fast should the car be traveling just as it leaves the cliff in order just to clear the river and land safely on the opposite side?

What is the speed of the car just before it lands safely on the other side?

Problem (cont)

\[
\begin{align*}
    v_x &= v_{x0} \\
    x &= x_0 + v_{x0}t \\
    v_y &= -gt \\
    y &= y_0 - \frac{1}{2}gt^2
\end{align*}
\]

1. Use vertical motion to find time in the air (last equation)
2. Then find \( v_0 \) from the second equation
Problem 3.55

According to the Guinness Book of World Records, the longest home run ever measured was hit by Roy "Dizzy" Carlyle in a minor league game. The ball traveled 188 m (618 ft) before landing on the ground outside the ballpark.

Assuming the ball's initial velocity was 45° above the horizontal and ignoring air resistance, what did the initial speed of the ball need to be to produce such a home run if the ball was hit at a point 0.9 m (3.0 ft) above ground level? Assume that the ground was perfectly flat.

How far would the ball be above a fence 3.0 m (10 ft) high if the fence was 116 m (380 ft) from home plate?

Problem (cont)

\[
\begin{align*}
\begin{array}{c}
\begin{align*}
x_0 &= 0 \\
y_0 &= 0.9 \text{ m} \\
x_f &= 188 \text{ m} \\
y_f &= 0.0 \text{ m} \\
v_0 &= ?
\end{align*}
\end{array}
\begin{align*}
\begin{cases}
\frac{v_x}{v_0} = x_0 & \\
x = x_0 + v_x t & \\
v_y = v_{y0} - gt & \\
y = y_0 + v_{y0} t - \frac{1}{2} gt^2 & \\
1. \text{ find time for horizontal motion from the second equation}
\end{cases}
\end{align*}
\begin{align*}
2. \text{ using this time find } v_0 \text{ from the last equation}
\end{align*}
\begin{align*}
3. \text{ a fence – then find time to fly to the fence in } x \text{ direction and then } y \text{ position}
\end{align*}
\end{align*}
\]
**Problem 3.63**

A physics professor did daredevil stunts in his spare time. His last stunt was an attempt to jump across a river on a motorcycle. The takeoff ramp was inclined at $53.0^\circ$, the river was $40.0$ m wide, and the far bank was $15.0$ m lower than the top of the ramp. The river itself was $100$ m below the ramp. You can ignore air resistance.

a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank?

b) If his speed was only half the value found in A, where did he land?

---

**Problem 3.63**

Data

a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank?

b) If his speed was only half the value found in A, where did he land?

Given: $(x, x_0), (y, y_0), \alpha_0$

Unknown: $v_0, t$

a) solve the system

b) find vertical time (now $v_0$ is given) and then the horizontal distance
How far and how high?

The fastest man (100 meters): 9.74 s
Asafa Powell (Jamaica) Rieti, Italy September 9, 2007
How high can he jump if … (world record 2.45 m)
How far can he jump if … (world record 8.95 m)

\[ h = \frac{v^2}{2g} \]
\[ R = \frac{v^2}{g} \]

Going beyond of simple projectile motion

WWI – the largest cannon: effect of air resistance

<table>
<thead>
<tr>
<th>WWI cannon</th>
<th>speed 1600 m/s</th>
<th>angle 52°</th>
<th>drag coeff. C=0.06</th>
<th>air density 1.25 kg/m³</th>
<th>M_{\text{projectile}} = 94 kg</th>
<th>R_{\text{projectile}} = 0.1 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude (km)</td>
<td>range (km)</td>
<td>no air resistance</td>
<td>yes (same air density)</td>
<td>yes $\rho_0 \exp(-y/y_0)$</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>
Going beyond of simple projectile motion

effect of air resistance – the acceleration is no longer constant

\[ mg = \frac{1}{2} \rho A v^2 \]

<table>
<thead>
<tr>
<th>object</th>
<th>speed (m/s)</th>
<th>speed (mph)</th>
<th>distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>shot</td>
<td>145</td>
<td>316</td>
<td>2500</td>
</tr>
<tr>
<td>sky diver</td>
<td>60</td>
<td>130</td>
<td>430</td>
</tr>
<tr>
<td>baseball</td>
<td>42</td>
<td>92</td>
<td>210</td>
</tr>
<tr>
<td>basketball</td>
<td>20</td>
<td>44</td>
<td>47</td>
</tr>
<tr>
<td>raindrop</td>
<td>7</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>parachutist</td>
<td>5</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>
Part 3

Uniform Circular Motion

References

C. Lindsey et al The Physics and Technology of Tennis, USRSA (2004)
Motion in a circle

Finding motion information

- velocity change,
- average acceleration,
- and instantaneous acceleration may be found.
**Uniform Circular Motion**

Magnitude of the velocity remains constant, but direction of the velocity changes continuously.

Motion is accelerated.

P can be located independently by either Cartesian coordinates \((x, y)\) or polar coordinates \((r, \theta)\).

The easiest way is to use polar coordinates.

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**Unit vectors in polar coordinates**

Introduce unit vectors \(\hat{r}\) and \(\hat{\theta}\):

\(\hat{r}\) points in the direction of increasing \(r\) when \(\theta\) is held constant,

\(\hat{\theta}\) points in the direction of increasing \(\theta\) when \(r\) is held constant.

Recall that \(i\) and \(j\) are constant in both magnitude and direction, \(\hat{r}\) and \(\hat{\theta}\) are constant in magnitude, but they are not constant in direction.

As \(P\) moves with respect to time, these polar unit vectors change their directions.

Consequently, they have non-zero time derivatives.

We need to calculate those time derivatives.
**Time derivatives of unit vectors**

Express polar unit vectors in terms of Cartesian unit vectors:

\[
\begin{align*}
\hat{r} & = \hat{i} \cos \theta + \hat{j} \sin \theta \\
\hat{\theta} & = -\hat{i} \sin \theta + \hat{j} \cos \theta
\end{align*}
\]

Differentiating radial unit vector:

\[
\frac{d}{dt} \hat{r} = -\hat{i} \sin \theta \frac{d\theta}{dt} + \hat{j} \cos \theta \frac{d\theta}{dt}
\]

\[
= \frac{d\theta}{dt} (-\hat{i} \sin \theta + \hat{j} \cos \theta) = \frac{d\theta}{dt} \hat{\theta} \equiv \omega \hat{\theta}
\]

where \( \omega \) is called the **angular velocity**

---

**Time derivative of azimuthal unit vector**

Differentiating azimuthal unit vector:

\[
\frac{d}{dt} \hat{\theta} = -\hat{i} \cos \theta \frac{d\theta}{dt} - \hat{j} \sin \theta \frac{d\theta}{dt}
\]

\[
= -\frac{d\theta}{dt} (\hat{i} \cos \theta + \hat{j} \sin \theta) = -\frac{d\theta}{dt} \hat{r} = -\omega \hat{r}
\]

Summary: \( \frac{d\hat{r}}{dt} = \omega \hat{\theta} \), \( \frac{d\hat{\theta}}{dt} = -\omega \hat{r} \)

Differentiating position vector \( \vec{r} = r \hat{r} \) gives needed kinematical relation for velocity

\[
\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt} = \frac{dr}{dt} \hat{r} + r \omega \hat{\theta}
\]

Radial component of velocity is \( v_r = \frac{dr}{dt} \)

Tangential component of velocity is \( v_\theta = \omega r \)
**Kinematical relations for acceleration**

Differentiating velocity gives

\[
\ddot{\mathbf{r}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (v_r \hat{\mathbf{r}} + \omega r \hat{\mathbf{\theta}}) = \left( v_r \frac{dv_r}{dt} + i_r \frac{dv_r}{dt} \right) + \left( \omega r \frac{dr}{dt} + \omega \frac{d\omega}{dt} \right) r \frac{dr}{dt} + r \frac{d\omega}{dt} \frac{dr}{dt} i_\theta = \hat{v}_r (d\mathbf{v}_r dt - \omega^2 r \hat{\mathbf{r}}) + i_\theta \left( 2v_r \omega + r \frac{d\omega}{dt} \right)
\]

General expression for acceleration is quite complicated

It is simpler for motion on a circular path \((v_r = 0)\) at constant speed \((d\omega/dt = 0)\)

---

**Circular motion with constant speed**

- For motion in a circle at constant speed, \(r = \text{const} \) and \(\omega = \text{const}\)
  
  \[\Rightarrow \left\{ \begin{array}{l}
v_r = \frac{dr}{dt} = 0, \quad \frac{dv_r}{dt} = 0, \quad \frac{d\omega}{dt} = 0 \end{array} \right\} \Rightarrow \ddot{\mathbf{r}} = -\omega^2 r \hat{\mathbf{r}}
  \]

Now acceleration is totally in the radial direction and is oriented toward the center of motion

It is called centripetal acceleration.

An alternative way of expressing the centripetal acceleration:

\[
a_r = -\omega^2 r = -\frac{v^2_\theta}{r^2} r = -\frac{v^2_\theta}{r}
\]
Circular motion with constant speed

(a) Uniform circular motion

\[ a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \]

\[ T = \frac{2\pi r}{v} \]

Acceleration has constant magnitude but varying direction.

Velocity and acceleration are always perpendicular.

Part 4

Relative Motion of Inertial Frames
**Kinematical relations for acceleration**

Consider a frame \((x', y')\), with axes parallel to those of \((x, y)\) frame, and the origin of \((x', y')\) located at \(\vec{R} = \vec{u}t\),

\[
\vec{r}' = \vec{R} + \vec{r}'
\]

where \(\vec{u}\) is the constant velocity of the primed frame relative to the unprimed frame.

Such frames are called inertial frames of reference.

**Relations between two inertial frames**

Note that

\[
\vec{r}' = \vec{R} + \vec{r}'
\]

Differentiate with respect to time to obtain

\[
\vec{v} = \vec{u} + \vec{v}'
\]

Since \(\vec{u}\) is constant, one further time-differentiation yields

\[
\vec{a} = \vec{a}'
\]

This result will be particularly important when we will consider the principles of Newtonian dynamics.