Modern Physics - Problem Set 2 – Solutions

Problem 1

Please answer the following questions with “Y” or “N”:

1a) Y
Light can travel $ct = 1500 \text{ m} = 1.5 \text{ km}$ in $5 \mu \text{s}$, so nothing can travel from one flash to the other. Another way of saying this is that the invariant interval is $ds^2 = (1.5 \text{ km})^2 - (3 \text{ km})^2 = -6.75 \text{ km}^2$ which is negative and therefore space-like.

1b) N
Since the invariant interval is the same for all observers, it's always negative = space-like.

1c) Y
Since they are separated by a space-like interval, their ordering depends on the coordinate system. In a system that is moving very fast towards the “later” (according to me) light flash, it will occur before the “earlier” one.

Problem 2

2a) 1 – Both S and S’ measure the same speed for a ball moving in the x-direction: No, see velocity addition rule
2 – Both S and S’ measure the same length for a pole (stationary in S) that is pointing in the y-direction: Yes, since the y-direction is perpendicular to the relative motion
3 – Both S and S’ measure the same speed for a ball moving in the y-direction: No, see velocity addition rule
4 – Both S and S’ measure the same velocity vector for a light wave moving in the z-direction (i.e. the same three components of that vector): No, see velocity addition rule (the individual components $u_x$, $u_z$ of the light ray change, only the overall speed doesn’t)

2b) 4
1 – According to an observer on Earth, this event is in the absolute future of the origin. Yes
2 – For any observer, the event has a time-like separation from the origin. Yes
3 – The invariant interval $\Delta s^2$ between this event and the origin, according to Earth, is $1 \text{ m}^2$. Yes
4 – The event happens at $ct = 1 \text{ m}$, according to Earth’s clock. No, it would be “time-dilated” to $\gamma \text{ m}$.

2c) 3
1 – The stick appears to have a length of only $8.72 \text{ m}$ in S. Yes
2 – At some point in time, the stick is observed (according to careful measurements in S) to be behind the building in its entire length. Yes
3 – The stick does not have a “true” (rest)length – it is completely arbitrary which coordinate system one uses to measure its length, and every coordinate system will give a different but equally valid answer. No: the rest length of the stick (the length as measured in the coordinate system in which it is at rest) is “privileged” over all other coordinate systems and can be called its “true” length.
4 – From the point of view of S’, the building is way too short to obscure the stick at any time. Yes (it’s contracted to much less than $10 \text{ m}$).
Problem 3

Two supernovae go off in our “cosmic neighborhood” – one at the stroke of midnight on December 31, 2020, and the second exactly 80 years later. Careful measurements show that the first supernovae was located at $x = -20$ ly, $y = 0$ and $z = 0$ in Earth’s coordinate system (S), while the second was located at $x = +20$ ly, $y = 30$ ly and $z = 0$.

3a) What is the invariant interval $\Delta s^2$ between the two supernova explosions?

$$(80 \text{ ly})^2 - (40 \text{ ly})^2 - (30 \text{ ly})^2 = 3900 \text{ ly}^2$$

3b) How fast would a spaceship have to travel (relative to Earth) to be present at BOTH explosions? (Never mind that the first one would destroy it…)

The spatial distance is 50 ly, so to cover this in 80 years requires a speed of 0.625 c

3c) How much time would have elapsed, according to that spaceship’s clock, between the two explosions?

This is directly given by the square-root of the invariant interval: 62.45 years (less than 80 years due time dilation)

Problem 4

Show algebraically that if you have two numbers $u$ and $v$ that are both inside the interval $[-1…1]$, then the expression $\frac{u+v}{1+uv}$ can never have a magnitude larger than 1. Explain how this proves that an object moving with velocity smaller than or equal to the speed of light in ONE inertial system can never move with a velocity greater than the speed of light relative to any other inertial system.

Answ.: We have to show that

$$\left| \frac{u+v}{1+uv} \right| \leq 1 \iff \left( \frac{u+v}{1+uv} \right)^2 \leq 1 \iff u^2 + 2uv + v^2 \leq 1 + 2uv + u^2v^2$$

$$\iff u^2 + v^2 \leq 1 + u^2v^2 \iff v^2 - u^2v^2 \leq 1 - u^2 \iff (1 - u^2)v^2 \leq 1 - u^2$$

The last inequality is definitely true if $u = \pm 1$ (both sides are zero), and if $|u| < 1$, we can divide by the positive quantity $(1 - u^2)$ on both sides and get $v^2 \leq 1$ which is true by assumption. So the last expression is always true under our assumptions and therefore the first one is, as well.

Of course, if we replace $u$ by $u_{x'}/c$ and $v$ by $v/c$, we get the known relationship for the velocity

$$\frac{u_{x'}}{c} = \frac{u_{x'}/c + v/c}{1 + u_{x'}v/c^2}$$

in system S for an object moving with velocity $u_{x'}$ in S’ (with S’ moving with $v$ in the x-direction relative to S), i.e. the rule for parallel velocity addition. Our result then shows that going from one coordinate system to another can never result in a speed larger than the speed of light, c.