Examples

1) Free particles

\[ H = \frac{p^2}{2m} \Rightarrow \psi_p(x,t) = A e^{i\frac{p}{\hbar} x - i \frac{p^2}{2m} t} \]

is a possible solution of \[ i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi \]

In fact, there is such a solution for every value of \( p \ (-\infty, \infty) \). This makes sense because classically, if we know \( p \) we know \( E = \frac{p^2}{2m} \), and of course \( \psi_p \) is an ES to \( H \) with eigenvalue \( p \) for each \( t \).

a) Important point: This particular form of a solution to S.E. is called "separable", since the dependence of \( \psi_p \) on \( x \) is "separate" from its dependence on \( t \) - it's just the product of 2 different functions, one of \( x \) (only) and one of \( t \) (only): \[ \psi_p(x,t) = \psi_p(x) \cdot e^{-\frac{iE\hbar t}{\hbar}} \text{ with } E = \frac{p^2}{2m} \]

Such solutions are especially important, since the spatial part is an ES to \( H \) with eigenvalue \( E \). So we solved S.E. in 2 parts: i) first, find solution to \( H \psi(x,t) = E \psi(x,t) \)

ii) Write full solution as \( \psi(x,t) = \psi(x) e^{-\frac{iE\hbar t}{\hbar}} \)

Clearly, such solutions (often called "stationary" which only makes sense for bound states) play an especially important role.
iii) They are states with **DEFINITE** ("*sharp*")
values of energy, \( E \) (since they are eigenvectors to \( H \)).
If we prepare a system in such a state, we will know with certainty what energy we will measure, \( E \).

iv) *Expectation values of all operators* (that act only on the \( x \)-coordinate and are time-independent) will be constant with time:
\[ \int_{-\infty}^{\infty} \psi^*(x,t) \hat{O} \psi(x,t) \, dx = \int_{-\infty}^{\infty} \sum_{n} \psi_{E}^*(x) \, \hat{O} \psi_{E} (x) \, e^{-\frac{iE_{n}t}{\hbar}} \, dx \]

-> "Nothing ever happens", the state is totally frozen ("stationary"). These states tend to be stable (at least stable enough to observe) because they do not emit radiation (except when coupled to an external field \( \to \) no longer eigenstates). In particular, 
\( E = \text{const.} \) and "*sharp*". If we find the ES to \( H \) with the lowest possible \( E \), we have the "ground state" which should be absolutely stable (energy conservation).

v) Of course not all solutions to S.E. look like that (are separable), but *(THEOREM)* we can build up all possible solutions as linear superpositions of such solutions:
\[ \psi(x,t) = \sum_{E} C_{E} \psi_{E}(x) \, e^{-\frac{iE_{n}t}{\hbar}} \, dx \]
We realize that in our special case (free particle), the separable solution is NOT a "real" solution, since \( \psi_p(x) = e^{ipx} \) is not normalizable (not a "real state")! But we can build up all true solutions from these special ones:

\[
\psi(x,t) = \int_{-\infty}^{\infty} \psi_p(x) e^{iEt} e^{-ipx} dp \quad \text{w/} \quad E_p = \frac{p^2}{2m}
\]

**Special case:** Gaussian distribution \( \psi_p(x) = \frac{1}{\sqrt{2\pi \sigma_p^2}} e^{-\frac{(x - \mu_p)^2}{2\sigma_p^2}} \)

\( \to \text{Gaussian wave packet that moves along } x \) with average momentum \( \mu_p \).

\( \to \) if \( \sigma_p \) is small:

\[
\begin{array}{c}
\text{very similar to} \\
\text{classical particle} \\
\text{with fixed momentum } \mu_p
\end{array}
\]

\( \H(x,p) \) happens that \( \mu_p \approx \frac{\hbar}{2\sigma_p} \) (see below)

\( \Rightarrow \sigma_x = \frac{\hbar}{2\sigma_p} \quad \text{the closer we get to a true ES of } \H, \text{ the less well we know position, and vice versa.} \)

**Example:** 1 μg dust speck moving with \( 1000 \pm 0.001 \text{ nm/s} \) (10^6 precision) \( \Rightarrow \sigma_p = 10^{-13} \text{ kg m/s} \) \( \Rightarrow \Delta x = \frac{\hbar}{2\sigma_p} \approx 0.5 \times 10^{-36} \)

\( \hbar = 6.63 \times 10^{-34} \text{ Js} \) \( \Rightarrow \) reason classical mechanics (point particle w/ known position & momentum) works.

Electron: \( m = 10^{-30} \text{ kg} \), assume \( \Delta v = 1000 \text{ m/s} \) \( \Rightarrow \Delta x = 50 \text{ ång} \)

Note: over time, \( \sigma_x \) increases b/c. of \( \sigma_p \) \( \Rightarrow \) spreading out of \( \psi_p \).

Now we apply this method of solving SE to some BOUND states.
2) well with \( \infty \) high walls \( (\text{Tipler 237-245}) \)

Simplest case of a binding potential

\[
V(x) = \begin{cases} 
0, & 0 \leq x \leq L \\
\infty, & \text{else} 
\end{cases}
\]

Classical solutions: pick arbitrary initial speed, position \( \Rightarrow \) keeps bouncing back and forth forever.

Qn.1: How to solve \( H \psi_E = E \psi_E \) with \( H = \frac{p^2}{2m} + V(x) \)

\[
\Rightarrow a) \text{ for } x < 0, \: x > L: \: V(x) = \infty \\
\left\{ \begin{array}{c}
\text{cannot work unless } V_E(x) = 0 \text{ there (V(x) p(x) blows up,}
\end{array} \right.
\]

\[
\Rightarrow b) \text{ for } 0 \leq x \leq L: \: \text{equal to FREE particle!}
\]

Solutions: \( \psi_{\text{free}} = A e^{i \frac{p}{\hbar} x} \) for any \( p \), \(-\infty < p < \infty\)

\[ \text{BUT: need continuous solution at } x = 0, L! \]

Can't work with just \( \psi_p \). Note: classical bouncing back and forth \( \Rightarrow \) maybe need to combine 2 \( \psi \)'s with \( p \) and \(-p \).

Yes! if we pick

\[
\left( \frac{1}{i} e^{i \frac{p}{\hbar} x} - e^{-i \frac{p}{\hbar} x} \right) \Rightarrow \sin \frac{p}{\hbar} x \Rightarrow 0 \text{ for } x = 0!
\]

\[ \text{BUT: also need } \psi_p \rightarrow \sin \frac{p}{\hbar} L = 0! \]

\[ \Rightarrow \text{cannot accept all possible } p, \text{ only those with} \]

\[ \frac{pL}{\hbar} = n \pi \]

\[ p_n = \frac{n \pi \hbar}{L} \]

\[ \Rightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad n = 1, 2, ... \]

\[ \text{Energy is quantized!} \]

[Diagram showing energy levels: \( E_1, E_2 \) with \( L \)]

\[ \text{Lowest ground state } E \text{ is NOT } 0! \]

Discuss features of solutions, in particular \( |\psi(x)|^2 \), \( \langle x \rangle, \langle p \rangle, \langle p^2 \rangle \), etc.
3) Harmonic Oscillator (Tippler 253 - 257
and online: Schrödinger's trick)

Physicists LOVE the HO. It's probably the only bound state system that you have really solved in PHYS 231 (planetary orbits are so much harder, so probably the solution was just handed down to you). There are very good reasons for this love:

a) HO is "simple" (at least in CM)

b) Oscillations are (some of the) most fundamental features of the world -> basis for all waves

c) Most realistic potentials look like HO, close to equilibrium point:

\[ V(x) = V(x_0) + \frac{dV}{dx}(x_0)(x - x_0) + \frac{1}{2} \frac{d^2V}{dx^2}(x_0)(x - x_0)^2 \]

= 0 because force = 0 @ equilibrium.

compare to mass on spring:

\[ F = -kx \quad \Delta W = \int_{0}^{x} F \, dx = -\frac{1}{2}kx^2 \]

\[ V(x) = -\Delta W = \frac{1}{2}kx^2 \quad (x = 0 \text{ is equilibrium here!}) \]

Solution: \[ m\ddot{x}(t) = -kx(t) \Rightarrow x(t) = A \sin \omega t + B \cos \omega t \]

with \[ \omega^2 = \frac{k}{m} \]

Hamiltonian: \[ \frac{p^2}{2m} + \frac{1}{2}m \dot{x}^2 = \frac{1}{2m} p^2 + \frac{m \omega^2}{2} x^2 \]

Stationary Solution: \[ \frac{1}{2m} (-\hbar^2 \frac{\partial^2 \psi_E}{\partial x^2} + \frac{m \omega^2}{2} x^2 \psi_E(x)) = E \psi_E \]

Looks scary! Won't be able to solve in general (but: see "Schrödinger's trick" in online companion to Tippler's book -> pocketbook / weblink)

But: We can make it look a bit more symmetric in \( x \):

divide both sides by \( \hbar \omega \) \[ \Rightarrow -\frac{1}{2} \frac{1}{\hbar^2} \frac{\partial^2 \psi_E}{\partial x^2} + \frac{1}{2m \hbar^2} \frac{\partial^2 \psi_E}{\partial x^2} + \frac{E}{\hbar \omega} \psi_E = 0 \]

Call \( y = x \sqrt{\frac{m \omega}{\hbar}} \) \[ \Rightarrow -\frac{1}{2} \frac{\partial^2 \psi_E}{\partial y^2} + \frac{1}{2} y^2 \psi_E = \frac{E}{\hbar \omega} \psi_E \]

Strategy: Solve for \[ \psi_E(y) \text{, then replace } \]

\[ y \rightarrow x \sqrt{\frac{m \omega}{\hbar}} \]
Possible solutions? (Ask for suggestions)

a) \( e^{ikx} \rightarrow \text{no, don't fall off at large } |x| \)
   \( \Rightarrow \text{must find something that falls off a lot faster than } \frac{1}{x^2} \)
   to keep \( V \phi \) finite!

b) \( e^{-k|x|^2} \): only works for \( x > 0 \)
   \( e^{-k|x|^2} \): unpleasant thing at \( x = 0 \) (whereas solutions should be smooth in \( \frac{\partial^2}{\partial x^2} \))

c) \( \Rightarrow \text{Maybe Gaussian? Why not } \Rightarrow \text{solution SHOULD be well-localized (not go on to large } x) \)
   
   Let's try it: \( \psi_E(y) = e^{-\alpha y^2} \Rightarrow \frac{\partial^2 \psi}{\partial y^2} = -2\alpha y e^{-\alpha y^2} \)
   \( \Rightarrow \frac{\partial^2 \psi}{\partial y^2} = (-4\alpha y^2 - 2\alpha) e^{-\alpha y^2} = \alpha^2 e^{-\alpha y^2} - 2\alpha e^{-\alpha y^2} \)
   \( \Rightarrow \text{compare: works (only!) if } \alpha = \frac{1}{2} \text{ and } \alpha = \frac{1}{2} \)
   \( \Rightarrow \text{found solution: } \psi_E(x) = A \cdot e^{-\frac{\hbar \omega}{2m} x^2} \quad E = \frac{1}{2} \hbar \omega \)
   \( \Rightarrow \text{Gaussian wave packet again; but NO motion (centered at } x = 0) \)
   and NO "spreading" (\( \sigma_x = \frac{\hbar}{\sqrt{2mE}} \text{ for all time).} \)

Some bookkeeping

a) Normalization:
   \[ |\psi_E(x)|^2 = A^2 \cdot e^{-\frac{\hbar \omega}{2m} x^2} \Rightarrow A^2 \cdot \int_{-\infty}^{\infty} e^{-\frac{\hbar \omega}{2m} x^2} dx = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1 \Rightarrow A = \sqrt{\frac{1}{\sqrt{2\pi}}} \]

b) Full solution: \( \psi(x,t) = \psi_E(x) e^{-i\frac{\hbar \omega}{2m} t} \)
   \( \text{unically, only } \frac{1}{\sqrt{2\pi}} \text{ of choice} \)

c) Is this the ground state? YES! We already know that the Gaussian distribution is the ONLY example where
   \( \sigma_x \cdot \sigma_p = \frac{\hbar}{2} \) since \( H = \frac{\hbar \omega}{2} x^2 + 6p^2 \), \( \langle H \rangle = \frac{\hbar \omega}{4} + 6 \sigma_p^2 \)
   To lower \( E \) (and therefore \( \langle H \rangle \)) would require to lower either \( \sigma_x \) or \( \sigma_p \Rightarrow \text{but the other one would blow up!} \)
d) Properties of the solution:

i) Centered maximum at origin (different from classical oscillator which has highest probability at turning point)

ii) Classical turning points:
- \( E = \frac{1}{2} kx^2 \)
- \( \Rightarrow x_{\text{max}} = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2E}{m\omega^2}} \)

Solution:
- \( E = \frac{1}{2} mh \)  \( \Rightarrow x_{\text{max}} = \sqrt{\frac{m}{h}} \)
- Which is \( \sqrt{\frac{1}{2}} \), \( \sqrt{x} \) \( \approx \) not unreasonable.

But: \( |\psi_E(x)|^2 \neq 0 \) even for \( x \gg x_{\text{max}} \)!

Typical feature of QM: "leaking out" or "tunneling". E.g., if potential stops at \( x_{\text{max}} \) and falls back to zero

for \( |a| > x_{\text{max}} \), the classical oscillator would never "know" and stay confined in region \( -x_{\text{max}} \leq x \leq x_{\text{max}} \) "forever". QM oscillator can "leak out" or tunnel through energy barriers and then become a free particle within a finite time!

Example: \( \alpha \)-decay, fission
There are more eigenstates / energy eigenvalues, \( E_0 + h\omega \), \( E_0 = \frac{1}{2} h\omega \).

Remember: \( \hat{H} = \frac{1}{2} \left( -\frac{\partial^2}{\partial y^2} + \frac{1}{2} y^2 \right) \Rightarrow \hat{H} \Psi_{E_0} = E_0 \Psi_{E_0} = \frac{1}{2} \Psi_{E_0} \)

Let's try \( \Psi'(y) = y \cdot \Psi_{E_0}(y) \Rightarrow \hat{H} \Psi' = \frac{1}{2} \frac{\partial^2}{\partial y^2} y \Psi_{E_0} + \frac{1}{2} y^2 \Psi_{E_0} \)

\[
- \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} \frac{\partial}{\partial y} y \Psi_{E_0} + y^2 \Psi_{E_0} \right) + \frac{1}{2} y^2 \Psi_{E_0} = -\frac{1}{2} \frac{\partial^2}{\partial y^2} y \Psi_{E_0} + \frac{1}{2} y^2 \Psi_{E_0} = \frac{1}{2} y \left( \Psi_{E_0} \right)
\]

\( E_{E_0} = \frac{1}{2} y \left( \frac{1}{2} y \Psi_{E_0} + \frac{1}{2} y^2 \Psi_{E_0} \right) \rightarrow \frac{1}{2} y \left( \frac{1}{2} y \Psi_{E_0} \right) = \frac{1}{2} y^2 \Psi_{E_0} \Rightarrow \)

\( \Rightarrow \hat{E} \hat{V} \hat{E} = \frac{1}{2} (E_0 + 1) \hat{h} \omega = \frac{3}{2} \hat{h} \omega \)

Interesting shape:

Flips sign for \( x \to -x \)

Zero (probably) at \( x = 0 \)

Wider distribution in \( x \)

\( \Rightarrow \) larger amplitude

Note: in general, could have also used \( \Psi' = -\frac{2}{\partial y} \Psi_{E_0} \) (same result)

In general, define \( a^+ = y - \frac{2}{\partial y} \)

\[
\hat{H}(a^+ \Psi_E) = y \hat{H} \Psi_E - \frac{2}{\partial y} \Psi_E - \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} \Psi_E \right) + y^2 \frac{2}{\partial y} \Psi_E
\]

\[
y^2 \frac{2}{\partial y} \Psi_E = \frac{2}{\partial y} y^2 \Psi_E - 2y \Psi_E
\]

\[
\frac{\partial^2}{\partial y^2} \Psi_E + y \Psi_E \Rightarrow \hat{H}(a^+ \Psi_E) = (y - \frac{2}{\partial y}) \hat{H} \Psi_E + (y - \frac{2}{\partial y}) \Psi_E = a^+ (\hat{E} + 1) \Psi_E
\]
It follows that if $\psi_E$ is eigenstate with $EV = \varepsilon$, then $a^+\psi_E$ is ES $\uparrow$ EV $\varepsilon + 1 \Rightarrow a^+$ "increases energy by 1 tw" $\Rightarrow$ ladder operator.

One can generate all eigenstates using repeatedly $(a^+)^n \psi_E$, just need to normalize.

**General Higher energy eigenstates:**

0: $e^{-\frac{1}{2}y^2}$

1: $y e^{-\frac{1}{2}y^2}$

2: $(2y^2 - 1) e^{-\frac{1}{2}y^2}$ etc.

With energies $\frac{1}{2} \text{tw}$, $\frac{3}{2} \text{tw}$, $\frac{5}{2} \text{tw}$ etc. (equidistant -

any 2 states are $\Delta E = \text{tw}$ apart).

Polynomials times exponent $-\frac{1}{2}y^2$ (Gaussian)

$\Rightarrow$ called "Hermite polynomials"