1-D Translations

Consider the operator

$$T(\Delta x) |x\rangle = |x + \Delta x\rangle$$

Obviously this operator represents a translation in the $x$ direction by some distance $\Delta x$.

For an infinitesimal shift, $\epsilon \to 0$, we would have $T(\epsilon) |x\rangle = |x + \epsilon\rangle$. Applying this translation operator to an arbitrary state vector, $|\psi\rangle$ yields

$$T(\epsilon) |\psi\rangle = |\psi'\rangle$$

In order for this operator to be useful, the following properties must be true:

- If $|\psi|^2 = 1$, then $|\psi'|^2 = 1$
- $T(\Delta x \to 0) \to 1$
- $T(\Delta x_1) T(\Delta x_2) = T(\Delta x_1 + \Delta x_2)$

From the first requirement we have

$$\langle \psi' | \psi' \rangle = \langle \psi | T^\dagger(\epsilon) T(\epsilon) | \psi \rangle = 1$$

Since this must be valid for ANY arbitrary state vector, it must be the case that $T$ is unitary, or $T^\dagger(\epsilon) T(\epsilon) = T(\epsilon) T^\dagger(\epsilon) = 1$.

Let’s assume that $T$ can be represented as a linear combination of the unit operator and some arbitrary operator $G$ such that

$$T(\epsilon) = 1 - \frac{i\epsilon}{\hbar} G$$

and

$$T^\dagger(\epsilon) = 1 + \frac{i\epsilon}{\hbar} G^\dagger$$

To find what $G$ is, let’s calculate $T^\dagger(\epsilon) T(\epsilon)$. Dropping terms with order higher than $\epsilon$ (since it is infinitesimally small anyway), we see that

$$T^\dagger(\epsilon) T(\epsilon) = \left( 1 + \frac{i\epsilon}{\hbar} G^\dagger \right) \left( 1 - \frac{i\epsilon}{\hbar} G \right)$$

$$= 1 + \frac{i\epsilon}{\hbar} G^\dagger - \frac{i\epsilon}{\hbar} G$$

$$= 1 + \frac{i\epsilon}{\hbar} (G^\dagger - G)$$

$\therefore G$ is Hermitian.
Now that we know $G$ is Hermitian, let’s examine the commutator between $T(\epsilon)$ and the $X$ operator:

$$XT(\epsilon)|x\rangle = X|x + \epsilon\rangle = (x + \epsilon)|x + \epsilon\rangle$$

$$T(\epsilon)X|x\rangle = T(\epsilon)x|x\rangle = x|x + \epsilon\rangle$$

So, a translation following by a measurement of the position yields a different result than first measuring the position followed by a translation (which should be no great shock).

$$[X, T(\epsilon)] = \epsilon T(\epsilon)$$

$$\left[ X, 1 - \frac{i\epsilon}{\hbar} G \right] = \epsilon \left( 1 - \frac{i\epsilon}{\hbar} G \right)$$

Again, we drop terms with order higher than $\epsilon$ and note that the unit operator commutes with anything.

$$[X, 1] - \frac{i\epsilon}{\hbar}[X, G] = \epsilon$$

$$\Rightarrow [X, G] = i\hbar$$

$$\Rightarrow G = P$$

Therefore, the generator for a translation is simply the momentum operator, and we have $T(\epsilon) = 1 - \frac{i\epsilon}{\hbar} P$.

All of these derivation was used on the assumption that the size of the translation, $\epsilon$, is infinitesimally small, but what if the desired shift is some finite distance $\Delta x$? In that case we break the translation up into $N$ small translations, apply the translation $N$ times, and allow $N$ to go to infinity.

$$T(\Delta x) = \lim_{N \to \infty} \left( T\left( \frac{\Delta x}{N} \right) \right)^N = \lim_{N \to \infty} \left( 1 - \frac{i}{\hbar} \frac{\Delta x}{N} P \right)^N = e^{-\frac{i\Delta x P}{\hbar}}$$
2-D Rotations

We can derive the operator responsible for 2-D rotations in much the same way that we derived the 1-D translation operator. First let’s note that, classically, a rotation through an angle $\varphi_0$ can be expressed using the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \varphi_0 & -\sin \varphi_0 \\ \sin \varphi_0 & \cos \varphi_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We define the operator $U[R_z(\varphi_0)]$ (causes a rotation through an angle $\varphi_0$ around the z axis) where

$$U[R_z(\varphi_0)] |\psi\rangle = |\psi_R\rangle$$

It would be very odd to have a rotation operator that didn’t rotate a position vector in the same way as a classical system. So, we must require that

$$U[R_z(\varphi_0)] |x,y\rangle = |x \cos \varphi_0 - y \sin \varphi_0, x \sin \varphi_0 + y \cos \varphi_0\rangle = |R\vec{r}\rangle$$

Using the same arguments as with the 1-D translation operator, we let $U[R_z(\varphi_0)] = 1 - \frac{i\varphi_0}{\hbar} G$. Now consider an infinitesimal rotation $\epsilon$:

$$U[R_z(\epsilon)] |x,y\rangle = |x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon\rangle$$

$$= |x - \epsilon y, y + \epsilon x\rangle$$

$$= T_x(-\epsilon y)T_y(\epsilon x) |x,y\rangle$$

$$= \left(1 - \frac{i(-\epsilon y)}{\hbar} P_x\right) \left(1 - \frac{i(\epsilon x)}{\hbar} P_y\right) |x,y\rangle$$

$$= \left(1 + \frac{i\epsilon y}{\hbar} P_x - \frac{i\epsilon x}{\hbar} P_y\right) |x,y\rangle$$

Since $[R_i, P_j] = \delta_{i,j}$, both $x$ and $y$ can be "promoted" to operators. We also note that this relationship is true for any vector $|x,y\rangle$, which allows us to relate the operators themselves. So we have

$$U[R_z(\epsilon)] = 1 - \frac{i\epsilon}{\hbar} (XP_y - YP_x) = 1 - \frac{i\epsilon}{\hbar} L_z$$
Rotation by a finite angle $\varphi_0$ can be obtained in a similar way to translating by a finite distance:

$$U[R_z(\varphi_0)] = e^{-i\varphi_0 L_z}$$

A very convenient coordinate system to use when working with this operator is polar coordinates. In polar coordinates, a rotation will only cause a change in the $\phi$ coordinate.

$$U[R_z(\varphi_0)] \ket{\rho, \varphi}_c = \ket{\rho, \varphi + \varphi_0}_c$$

Here, we introduce a new labeling for our basis vectors - note that they are still the same position eigenstates as before, just labeled with $(\rho, \varphi)$ instead of $(x, y)$. In fact, we simply define

$$\ket{\rho, \varphi}_c = \ket{x = \rho \cos \varphi, y = \rho \sin \varphi}.$$  

We can then introduce for any ket $\ket{\psi}$ its representation in these new variables as

$$\psi_c(\rho, \varphi) := \langle \rho, \varphi | \psi \rangle = \psi(\rho \cos \varphi, \rho \sin \varphi) = \langle x = \rho \cos \varphi, y = \rho \sin \varphi | \psi \rangle.$$  

Note that, by the laws of integration,

$$\int \int d\rho d\varphi \psi^*_c(\rho, \varphi) \psi_c(\rho, \varphi) = \int \int dx dy \psi^*(x, y) \psi(x, y) = 1$$

for proper normalization. This implies

$$\int \int d\rho d\varphi \ket{\rho, \varphi}_c \rho \langle \rho, \varphi \rangle = 1.$$  

For reference, we note the normalization of the new way of writing our basis vectors:

$$\langle \rho', \varphi' | \rho, \varphi \rangle_c = \langle \rho' \cos \varphi', \rho' \sin \varphi' | \rho \cos \varphi, \rho \sin \varphi \rangle = \delta(\rho' \cos \varphi' - \rho \cos \varphi) \delta(\rho' \sin \varphi' - \rho \sin \varphi).$$  

Using $\delta(f(x) - b) = \delta(x - f^{-1}(b))/|f'(x)|$, we can evaluate this expression as

$$\langle \rho', \varphi' | \rho, \varphi \rangle_c = \frac{1}{\cos \varphi'} \delta\left(\rho' - \rho \frac{\cos \varphi}{\cos \varphi'}\right) \delta\left(\rho \cos \varphi \tan \varphi' - \rho \sin \varphi\right)$$

$$= \frac{1}{\cos \varphi'} \delta\left(\rho' - \rho \frac{\cos \varphi}{\cos \varphi'}\right) \frac{\cos^2 \varphi'}{\rho \cos \varphi} \delta\left(\varphi' - \arctan(\sin \varphi/ \cos \varphi)\right) = \frac{1}{\rho} \delta(\rho' - \rho) \delta(\varphi' - \varphi).$$
To find a representation for \( \mathbf{L}_z \) in polar coordinates, consider an arbitrary wave function that has been rotated by an infinitesimal amount in polar coordinates:

\[
\psi_c(\rho, \varphi + \epsilon) = \langle \rho, \varphi + \epsilon | \psi \rangle
\]

\[
= \langle \rho, \varphi | \mathbf{U}[R_z(\epsilon)] | \psi \rangle
\]

\[
= \left\langle \rho, \varphi | 1 - \frac{i \epsilon}{\hbar} \mathbf{L}_z | \psi \right\rangle
\]

\[
= \psi_c(\rho, \varphi) + \frac{i \epsilon}{\hbar} \langle \rho, \varphi | \mathbf{L}_z | \psi \rangle
\]

We also note that

\[
\psi_c(\rho, \varphi + \epsilon) = \psi_c(\rho, \varphi) + \epsilon \frac{\partial}{\partial \varphi} \psi_c(\rho, \varphi) + \mathcal{O}(\epsilon^2)
\]

So,

\[
\frac{i}{\hbar} \langle \rho, \varphi | \mathbf{L}_z | \psi \rangle = \frac{\partial}{\partial \varphi} \psi_c(\rho, \varphi)
\]

\[
\rightarrow \langle \rho, \varphi | \mathbf{L}_z = -i \hbar \frac{\partial}{\partial \varphi} \langle \rho, \varphi |
\]

Now that we have a representation for \( \mathbf{L}_z \), it would be useful to know its related eigenvalues. If \( |l_z \rangle \) is an eigenfunction of \( \mathbf{L}_z \), then the related eigenvalue will be \( l_z \). Using the derivative form of \( \mathbf{L}_z \) will give

\[
-l_z \rho \frac{\partial}{\partial \varphi} \psi_{l_z}(\rho, \varphi) = l_z \psi_{l_z}(\rho, \varphi)
\]

\[
\rightarrow \psi_{l_z}(\rho, \varphi) = AR(\rho) e^{i l_z \varphi \hbar}
\]

To find \( l_z \) we note that \( l_z / \hbar \) must be an integer (since we require \( \psi(\rho, 2\pi) = \psi(\rho, 0) \)). So, \( l_z \) is quantized. More specifically,

\[
\frac{2\pi l_z}{\hbar} = 2\pi n
\]

\[
\rightarrow l_z = h n
\]