Graduate Quantum Mechanics - Problem Set 7 - Solution

Problem 1)
An atom of mass $4 \times 10^9 \text{ eV}/c^2$ has its position measured within 2 nm accuracy. Assume that it is in a Gaussian wave packet state afterwards. How much time will elapse before the uncertainty of our knowledge about its position has doubled? How about a 1 µg speck of matter that has been located to within 1 µm?

Answ.:
The momentum uncertainty will be $\Delta p = \hbar/2\Delta x = 50 \text{ eV}/c$. Given the mass, the velocity uncertainty is $\Delta p/m = 1.25 \times 10^{-8} c = 3.75 \times 10^9 \text{ nm}/s$ which means it will take about $\frac{1}{2}$ ns before the additional spread equals 2 nm. However, since widths have to be added in quadrature, it will be more like 1 ns before the position uncertainty has grown to 4 nm.

For the 2nd part, we need to express $\hbar$ in SI units. Plugging in the numbers gives a velocity uncertainty of $5.27 \times 10^{-20} \text{ m/s} = 5.27 \times 10^{-14} \mu \text{m}/s$. This yields $3.3 \times 10^{13}$ s for the uncertainty to grow to 2 µm, or about 1 million years.

Problem 2)
A point-like particle of mass $m$ sits in a one-dimensional potential well. The potential is infinitely high for $x < -s$ and for $x > +s$, while it is at a constant value of $V_0 > 0$ for $-s \leq x < 0$ and zero for $0 \leq x \leq s$. The particle is in the ground state (lowest energy eigenstate of the Hamiltonian) with energy $E_0 > V_0$.

Question: What is the probability that the particle can be found in the left half ($x < 0$) of the potential well?

Outline how you would solve this problem step by step, without actually solving the (transcendental) equations that you encounter:
1. Write down the one-dimensional Schrödinger equation for this problem.
2. Find the generic stationary solutions in the left and right half of the potential well (you may assume $E > V_0$).
3. List all boundary conditions that must be fulfilled (there are 4 of them!)
4. Rewrite your two half-solutions from item 2. above to explicitly fulfill as many of the boundary conditions as possible.
5. Outline how you would find the lowest energy (ground state eigenvalue $E$) that solves the one-dimensional Schrödinger equation. No closed algebraic solution is possible or required for this part - just explain which equation needs to be solved.
6. Assuming you have $E$, how would you determine the normalization constants for the two half-solutions?
7. Once you have those in hand as well, how can you answer the original question?

Answ.:
I got tired typing all this up – sorry! – so the answer is handwritten (see next pages)
1.) \(-\frac{b^2}{2m} \frac{\partial^2}{\partial x^2} \psi = (E - V(x)) \psi = \begin{cases} E - V_0, & -s \leq x < 0 \quad \text{I} \\ E, & 0 \leq x \leq s \quad \text{II} \end{cases}\)

2.) \(k_1 = \frac{\sqrt{2m(E - V_0)}}{\hbar}, \quad k_2 = \frac{\sqrt{2mE}}{\hbar}; \quad \text{then}\)

\[\psi_{\text{I}}(x) = A \sin k_1 x + B \cos k_1 x \quad (-s \leq x < 0)\]

\[\psi_{\text{II}}(x) = C \sin k_2 x + D \cos k_2 x\]

3.) \(\psi(-s) = 0\) \((\text{continuity}) \Rightarrow \text{can rewrite as}\)

\[-4.) \quad \psi_{\text{I}}(x) = A \sin (k_1 (x+s))\); \quad \text{similarly} \quad \psi(s) = 0 \Rightarrow \]

\[\psi_{\text{I}}(x) = C \sin (k_2 (x-s))\]

\((\text{continuity at} \quad x = 0 \Rightarrow A = \frac{C \sin k_1 s}{\sin k_1 s} = -C \frac{\sin k_2 s}{\sin k_2 s} \)

1st derivation \(\psi'_{\text{I}} = AK_1 \cos (k_1 (x+s)) = -k_1 C \frac{\sin k_1 s}{\sin k_2 s} \cos (k_1 (x+s))\)

\[\psi'_{\text{II}}(x) = k_2 C \cos (k_2 (x-s)) \quad \text{continuity} \]

\(-k_1 \cot k_1 s = k_2 \cot k_2 s\)

5.) \(\text{The equation must be solved by finding the lowest possible value of } E \text{ for which both sides are equal.}\)

\(\text{Since } k_2 > k_1 \text{ and because of the - sign, the solution will likely be for } \frac{\pi}{2} < k_2 s < \pi \text{ (at } k_2 s < 0)\)

\((\text{in this, writing dimensionless variables } \phi_{1,2} = k_{1,2} s)\)

\[k_1 = k_2 \sqrt{\frac{E - V_0}{E}} \Rightarrow \phi_1 = \phi_2 \sqrt{1 - \frac{V_0}{E}} \quad V_0 = \frac{2mV_0 s^2}{\hbar^2}, \quad \tau = \frac{E}{V_0}\]

\[\Rightarrow \phi_2 = k_2 s = \frac{2mV_0 s}{\hbar} \sqrt{\frac{E}{V_0}} = \sqrt{V_0} \cdot \tau \Rightarrow \]

\[-\sqrt{V_0} \cdot \tau \left( \sqrt{V_0} \cdot \tau \right) = V_0 \cdot \tau \cos \left( \sqrt{V_0} \cdot \tau \right) \text{ can be solved numerically for } \tau, \text{ given } V_0\]


6. There is only one unknown constant $C$.

Determine by normalizing the wave function:

$$
\int_{-\infty}^{\infty} |\psi|^2(x) \, dx = |C|^2 \left( \int_{-\infty}^{0} \frac{\sin^2 k_1 s}{\sin^2 k_1 s} \sin^2 (k_1 (x+s)) \, dx + \int_{0}^{\infty} \sin^2 (k_2 (x-s)) \, dx \right)
$$

$$
= |C|^2 \left( \frac{\sin^2 k_1 s}{\sin^2 k_1 s} \left[ \frac{S}{2} - \frac{1}{4k_1} \sin 2k_1 (x+s) \right]^{0}_{-S} + \frac{S}{2} - \frac{1}{4k_2} \sin 2k_2 (x-s) \right)
$$

$$
= |C|^2 \left( \frac{\sin^2 k_1 s}{\sin^2 k_1 s} \left[ \frac{S}{2} - \frac{1}{4k_1} \sin 2k_1 s + \frac{S}{2} - \frac{1}{4k_2} \sin 2k_2 s \right) \right)
$$

\[ \Rightarrow C = \left( \frac{\sin^2 k_1 s}{\sin^2 k_1 s} + 1 \right) \left( \frac{S}{2} - \frac{1}{2k_1} \sin^2 k_1 s \cot k_1 s \left( 1 - \frac{k_1^2}{k_2^2} \right) \right) \]

7. $P(x<0) = \int_{-\infty}^{0} |\psi|^2 \, dx = |C|^2 \frac{\sin^2 k_1 s}{\sin^2 k_1 s} \left( \frac{S}{2} - \frac{\sin 2k_1 s}{4k_1} \right)

\[
= \frac{\sin^2 k_1 s}{\sin^2 k_1 s} \left[ \frac{S}{2} - \frac{1}{2k_1} \sin^2 k_1 s \cot k_1 s \left( 1 - \frac{k_1^2}{k_2^2} \right) \right)
\]

\[
= \frac{\sin^2 k_2 s}{\sin^2 k_2 s} \left( 1 - \frac{\sin^2 k_2 s}{k_1 s} \cot k_1 s \left( 1 - \frac{k_1^2}{k_2^2} \right) \right)
\]

\[
= \frac{\sin^2 k_2 s}{k_1 s} \cot k_1 s \left( 1 - \frac{\sin^2 k_2 s}{k_1 s} \cot k_1 s \left( 1 - \frac{k_1^2}{k_2^2} \right) \right)
\]
Problem 3)

Consider the “Gaussian wave packet” from the lecture or p. 154 in Shankar. Calculate the probability current $j$, for every point $x$ at time $t = 0$. Using our result for the probability density, $\rho(x, t)$, show through explicit calculation (not by invoking general principles!!!) that the continuity equation for probability is fulfilled at time $t = 0$.

Answ.: 

$$\psi(x, t = 0) = \frac{1}{\sqrt{2\pi \sigma}} e^{i\frac{p_0 x}{\hbar}} e^{-\left(x-x_0\right)^2/4\sigma^2}$$

$$j(x, t = 0) = \frac{\hbar}{2mi \sqrt{2\pi \sigma}} \left( e^{i\frac{p_0 x}{\hbar}} e^{-\left(x-x_0\right)^2/4\sigma^2} \left[ i\frac{p_0}{\hbar} - \frac{x-x_0}{2\sigma^2} \right] e^{i\frac{p_0 x}{\hbar}} e^{-\left(x-x_0\right)^2/4\sigma^2} \right)$$

$$= \frac{i2p_0}{2mi \sqrt{2\pi \sigma}} e^{-\left(x-x_0\right)^2/2\sigma^2} = \frac{p_0}{m} \rho(x, t = 0); \frac{\partial j}{\partial x}(x, t = 0) = -\frac{x-x_0}{\sigma^2} \frac{p_0}{m} \rho(x, t = 0)$$

$$\rho(x, t) = \frac{1}{\sqrt{2\pi \left(\sigma^2 + \hbar^2 t^2 / 4m^2 \sigma^2\right)}} e^{-\left(x-x_0-p_0 t/m\right)^2 / 2(\sigma^2 + \hbar^2 t^2 / 4m^2 \sigma^2)}$$

$$\frac{\partial \rho(x, t)}{\partial t} = -\frac{1}{2\pi} \frac{2\hbar^2 t / 4m^2 \sigma^2}{\left(\sigma^2 + \hbar^2 t^2 / 4m^2 \sigma^2\right)^2} \rho(x, t) +$$

$$\left[ -2\frac{x-x_0-p_0 t/m}{\sigma^2 + \hbar^2 t^2 / 4m^2 \sigma^2} (-\frac{p_0}{m}) + \frac{(x-x_0-p_0 t/m)^2 2\hbar^2 t / 4m^2 \sigma^2}{2\left(\sigma^2 + \hbar^2 t^2 / 4m^2 \sigma^2\right)^2} \right] \rho(x, t)$$

$$\frac{\partial \rho(x, t)}{\partial t} \bigg|_{t=0} = \left(0 + \frac{p_0}{m} \frac{x-x_0}{\sigma^2} + 0\right) \rho(x, t = 0) = -\frac{\partial j}{\partial x}(x, t = 0)$$

q.e.d.