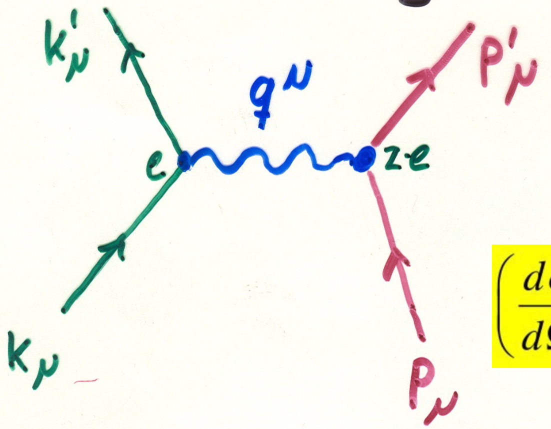
Form Factors

Elastic Scattering – Feynman diagram



Rutherford (classical)

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}$$

$$Q^2 := -q^\nu q_\nu$$

$$\mathcal{M}_{fi} = e\, j_\nu \left(-\frac{1}{q^\nu q_\nu}\right) z\, j^\nu$$

Based on $\Box^2 A_\nu = j_\nu \Rightarrow A_\nu = \left(\frac{1}{iq}\right)^2 j_\nu$

and $H_{int} = A_\nu j^\nu = V\rho - \vec{A}\cdot\vec{j}$

$$\Rightarrow \Delta\sigma = \underbrace{\frac{4z^2\alpha^2(\hbar c)^2}{Q^4}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)^{*)}}_{\sigma_M}\left(1+2\frac{\nu^2}{Q^2}\tan^2\frac{\theta}{2}\right)^{*)} \cdot \int d^3\vec{k}'\,\delta(E'-E'_{el})\left(= E'^2\cdot\Delta\Omega\right)$$

*) magnetic interaction due to electron spin

*) magnetic interaction due to target spin

Elastic Cross section – final form

$$\Delta\sigma = \frac{4z^2\alpha^2(\hbar c)^2}{Q^4}\cos^2\frac{\theta}{2}\left[\underbrace{\frac{G_E^2+\tau G_M^2}{1+\tau}}_{\text{LONGITUDINAL (charge)}} + \underbrace{2\tau\tan^2\frac{\theta}{2}\,G_M^2}_{\text{TRANSVERSE (magnetic)}}\right] \cdot E'^2\,\Delta\Omega\,\underbrace{\frac{E'}{E}}_{\text{Recoil}}$$

$\tau = \frac{\nu^2}{Q^2}$, $G_E(Q^2)$, $G_M(Q^2)$: Form Factors

Dirac Particle: $G_E = G_M = 1$ (const.)

Anomalous magnetic moment: $G_M \approx (1+\kappa)G_E$

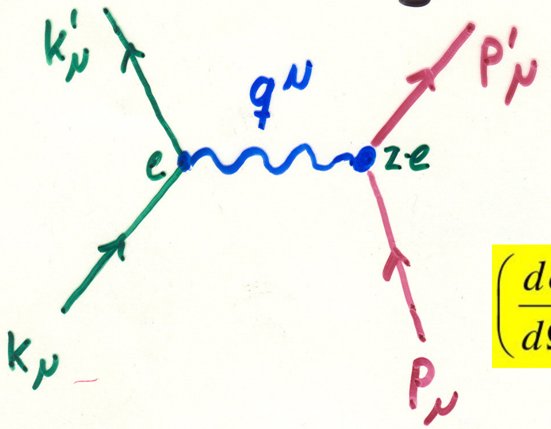
Extended Charge distribution:

$G_E(Q^2) \simeq$ Fourier transform of $\rho(r)$

Ex: $\rho(r) \simeq \frac{a^3}{8\pi}e^{-ar} \Rightarrow G_E(Q^2) \simeq \left(\frac{1}{1+\frac{Q^2}{a^2}}\right)^2$

(Dipole Form). $p: a^2 = 0.71\ \mathrm{GeV}^2$

Elastic Scattering – Feynman diagram



Rutherford (classical)

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}$$

$$Q^2 := -q^\nu q_\nu$$

$$\mathcal{M}_{fi} = e\, j_\nu \left(-\frac{1}{q^\nu q_\nu}\right) z\, j^\nu$$

Based on $\Box^2 A_\nu = j_\nu \Rightarrow A_\nu = \left(\frac{1}{iq}\right)^2 j_\nu$

and $H_{int} = A_\nu j^\nu = V\rho - \vec{A}\cdot\vec{j}$

$$\Rightarrow \Delta\sigma = \underbrace{\frac{4z^2\alpha^2(\hbar c)^2}{Q^4}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)^{*)}}_{\sigma_M}\left(1+2\frac{\nu^2}{Q^2}\tan^2\frac{\theta}{2}\right)^{*)}\cdot$$

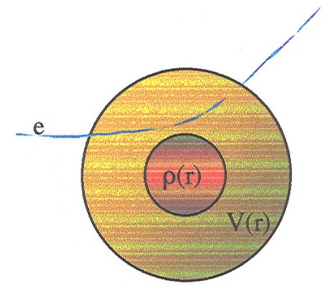
$$\cdot \int d^3\vec{k}'\,\delta(E'-E'_{el})\left(= E'^2\cdot\Delta\Omega\right)$$

*) magnetic interaction due to electron spin

*) magnetic interaction due to target spin

Form Factors

Low-medium energy: Distribution of charge and magnetism inside the hadron



$$\vec\nabla^2 V(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \quad \Rightarrow$$

$$q^2 V \propto \int e^{i\mathbf{qr}}\rho(\mathbf{r})d^3\mathbf{r} = F(q)$$

$$\mathcal{H} \approx -eV \propto \frac{F(q)}{q^2}$$

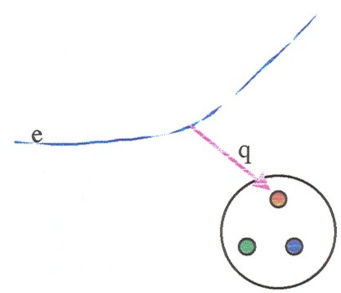
$$\frac{d\sigma}{d\Omega} \propto \left|\langle f|\mathcal{H}|i\rangle\right|^2 \propto \frac{F^2(q)}{q^4}$$

Ex: $\rho(r) = a\cdot e^{-\alpha r} \Rightarrow F(q^2) = (1+q^2/\alpha^2)^{-2}$ (Dipole Form)

$\alpha^2 = 0.71\ \mathrm{GeV}^2$

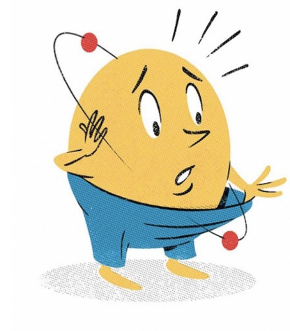
$$<r_E^2> \equiv -6\,\frac{dG_E(Q^2)}{dQ^2}\bigg|_{Q^2=0}$$

High energy: "Stability" of internal structure against hard "blows"



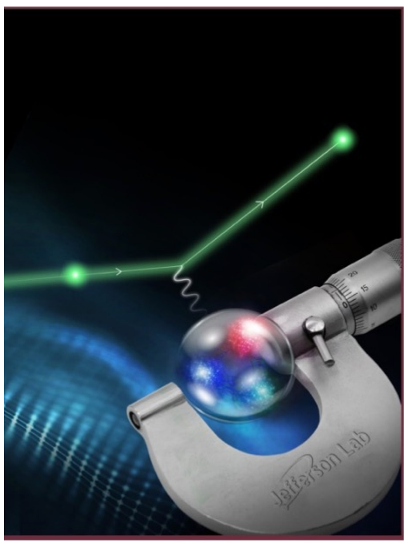
Can the hadron absorb a high momentum virtual photon without breaking apart?

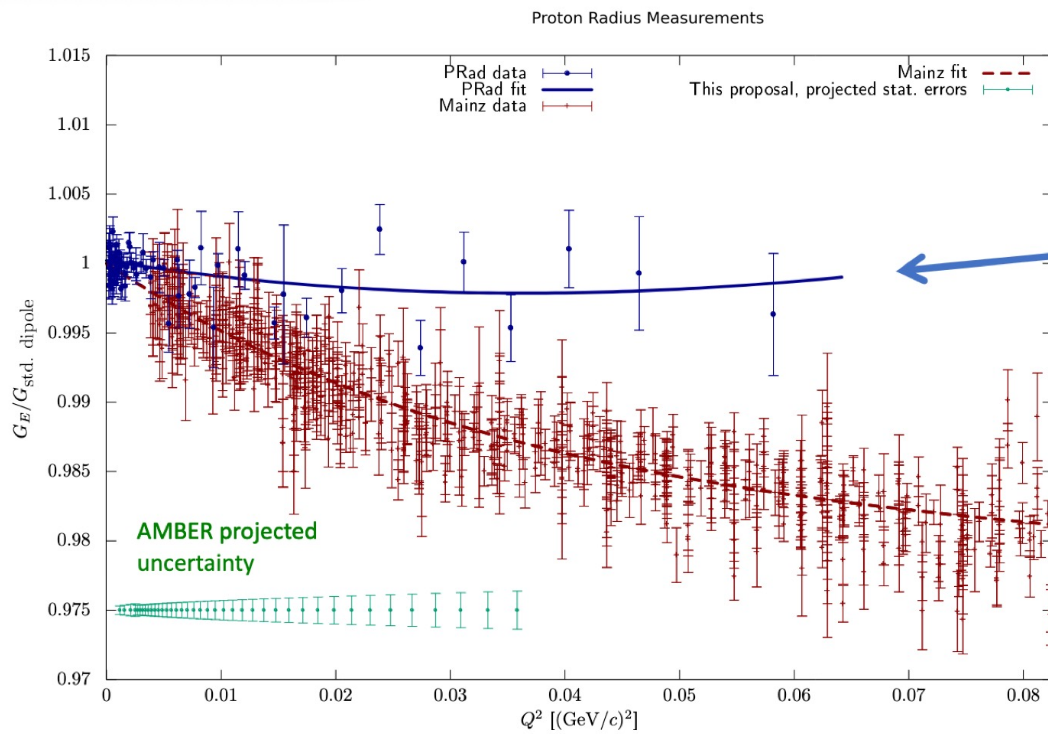
Proton Puzzle 1:



New York Times

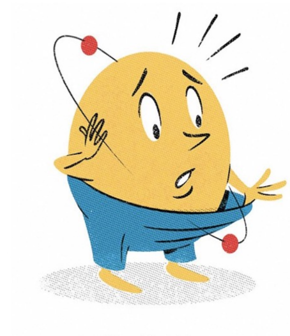
- Different ways of measuring proton radius disagree





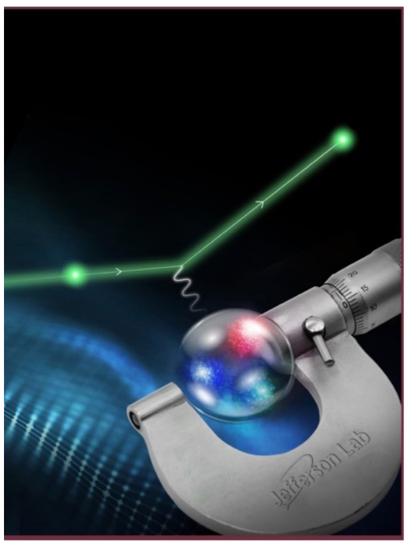
- Possible Explanation? Maybe people made mistaken assumptions in the past?

Proton Puzzle 1:



New York Times

- Different ways of measuring proton radius disagree



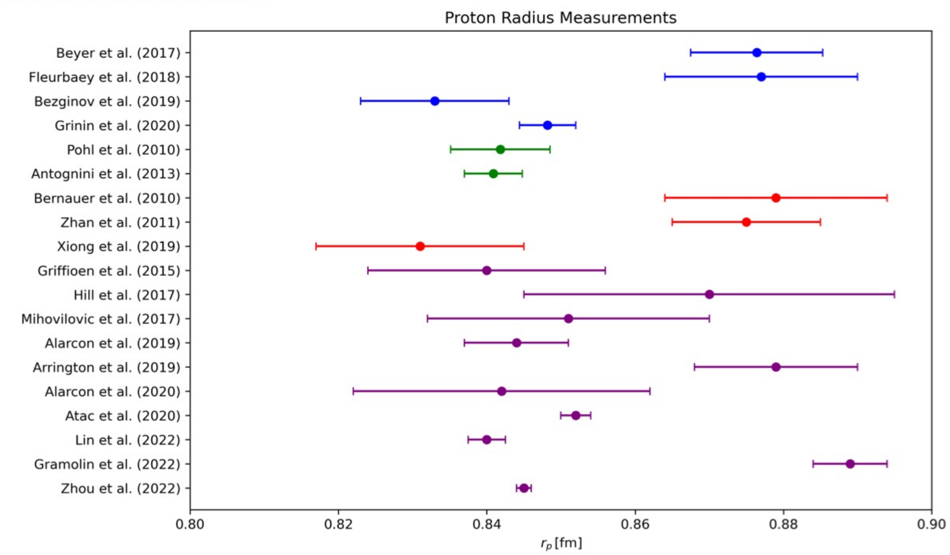


Figure 8: Proton radius measurements categorized by technique. The blue points represent results from Hydrogen Spectroscopy, the green points are from μ-Hydrogen Spectroscopy, the red points are from Ep Scattering, and the purple points are from Reanalysis of data.

- Possible Explanation? Maybe people made mistaken assumptions in the past?

Elastic cross section ($p'^2 = m^2$)

Recoil factor

Form factors

$$\frac{d\sigma}{d\Omega} = \sigma_M \frac{E'}{E}\left\{\left[F_1^2(Q^2) + \frac{Q^2}{4M^2}\kappa^2 F_2^2(Q^2)\right] + \frac{Q^2}{2M^2}[F_1(Q^2) + \kappa F_2(Q^2)]^2 \tan^2\frac{\theta}{2}\right\}$$

$$\boxed{= \sigma_M \frac{E'}{E}\left[\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} + 2\tau \tan^2\frac{\theta}{2} G_M^2(Q^2)\right]}$$

$$= \sigma_M \frac{E'}{E}\left[\frac{Q^4}{\vec{q}^4} R_L(Q^2) + \left(\frac{Q^2}{2\vec{q}^2} + \tan^2\frac{\theta}{2}\right) R_T(Q^2)\right]$$

Mott cross section

$$\sigma_M = \frac{\alpha^2 \cos^2\left(\frac{\theta_e}{2}\right)}{4E^2 \sin^4\left(\frac{\theta_e}{2}\right)} \longrightarrow \boxed{Q^4/E'^2}$$

F_1, F_2: Dirac and Pauli form factors

G_E, G_M: Sachs form factors (electric and magnetic)

$G_E(Q^2) = F_1(Q^2) - \tau\kappa F_2(Q^2)$ $\qquad$ $\tau = Q^2/4M^2$

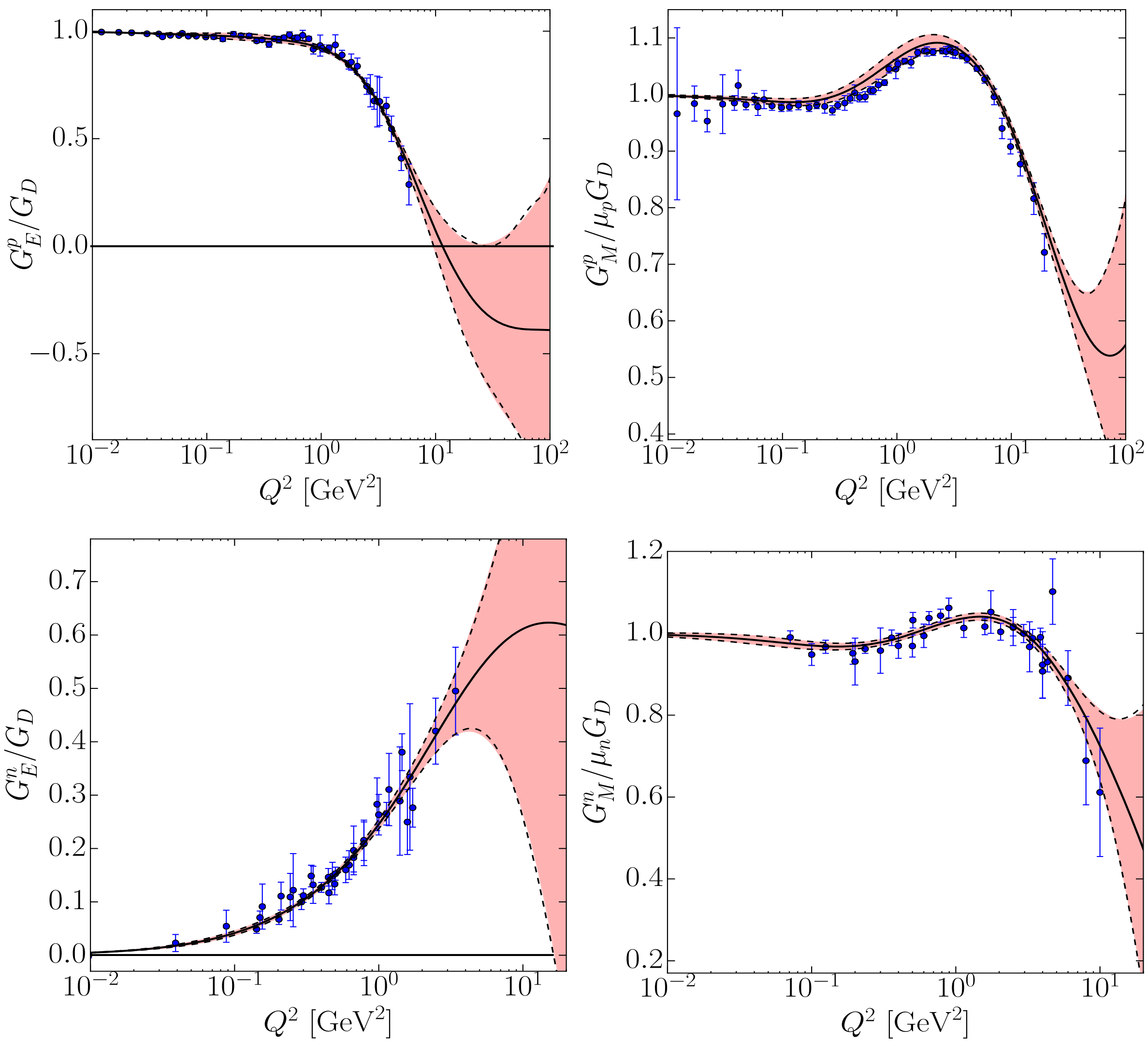
$G_M(Q^2) = F_1(Q^2) + \kappa F_2(Q^2)$ $\qquad$ κ = anomalous magnetic moment

R_L, R_T: Longitudinal and transverse response fn

Notes on form factors

- G_E, G_M, F_1 and F_2 refer to nucleons
  - $F_1^p(0) = 1$, $F_2^p(0) = \kappa_p = 1.79$
  - $F_1^n(0) = 0$, $F_2^n(0) = \kappa_n = -1.91$
  - $G_E^p(0) = 1$, $G_M^p(0) = 1 + \kappa_p = 2.79$
  - $G_E^n(0) = 0$, $G_M^n(0) = \kappa_n = -1.91$
- κ is the anomalous magnetic moment
- R_L and R_T refer to nuclei

Electromagnetic Form Factors

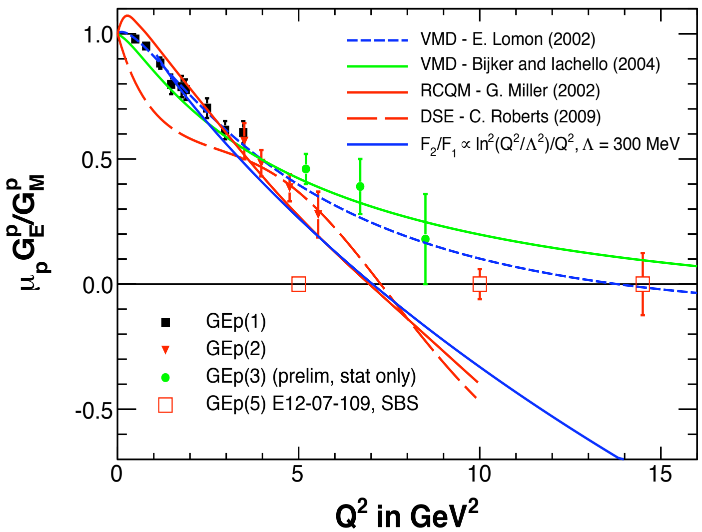


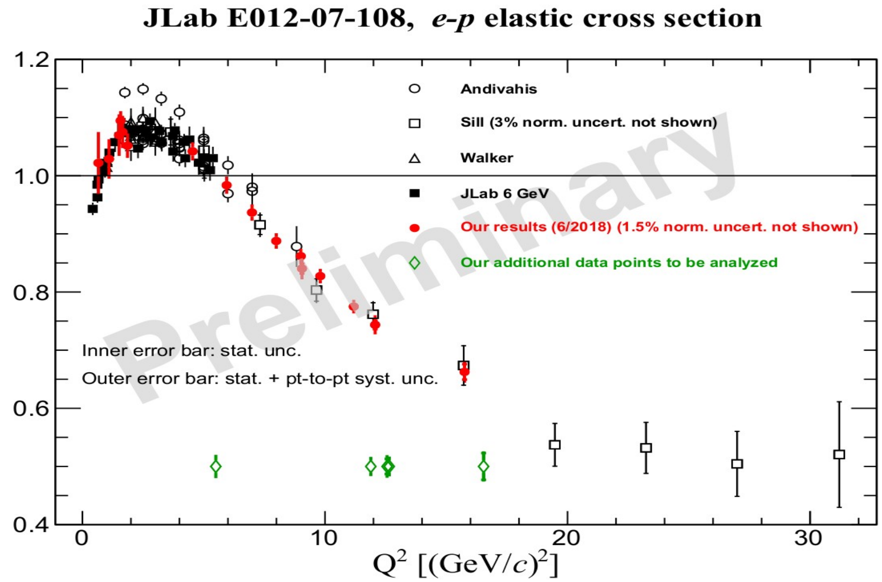
Z. Ye et al. / Physics Letters B 777 (2018) 8–15

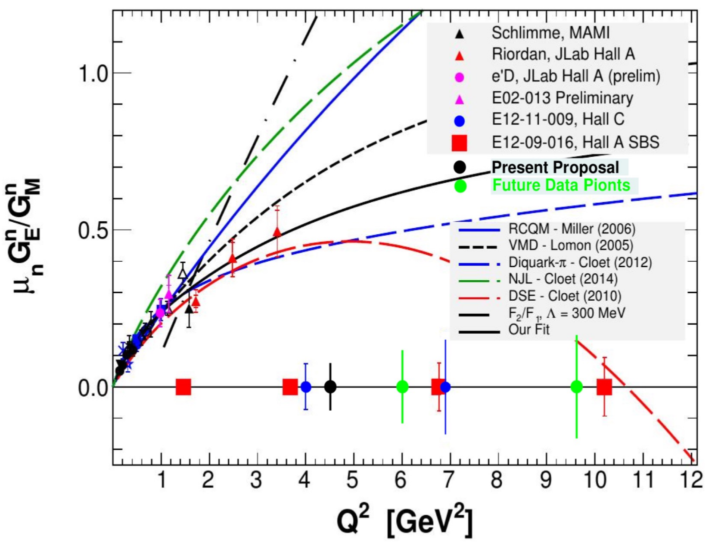
$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \frac{\epsilon (G_E^N)^2 + \tau (G_M^N)^2}{\epsilon(1+\tau)}$$

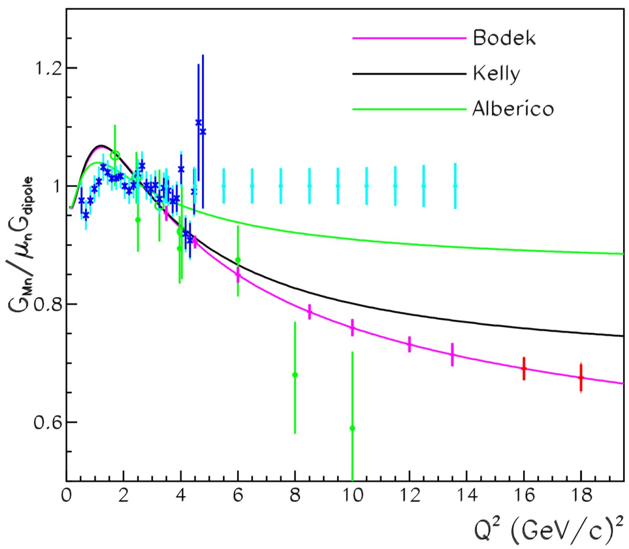
$$\tau = \frac{Q^2}{4m_N^2}, \quad \epsilon = \left[1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right]^{-1}$$

Electromagnetic Form Factors









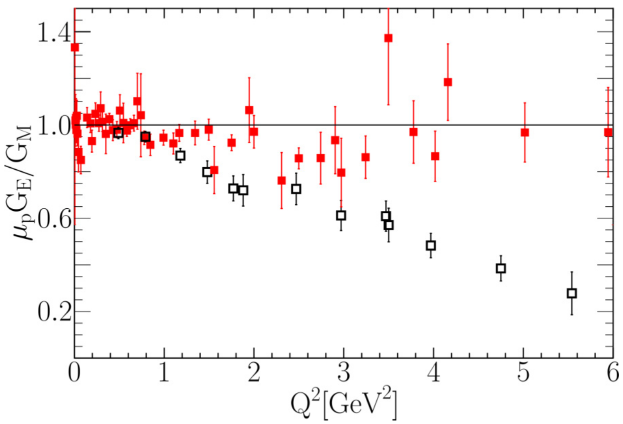
Z. Ye et al. / Physics Letters B 777 (2018) 8–15

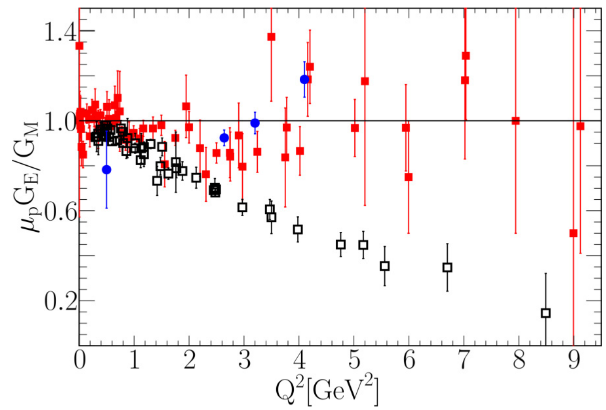
$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \frac{\epsilon(G_E^N)^2 + \tau(G_M^N)^2}{\epsilon(1+\tau)}$$

$$\tau = \frac{Q^2}{4m_N^2}, \quad \epsilon = \left[1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right]^{-1}$$

Proton Puzzle 2:

- Different ways of measuring G_E disagree

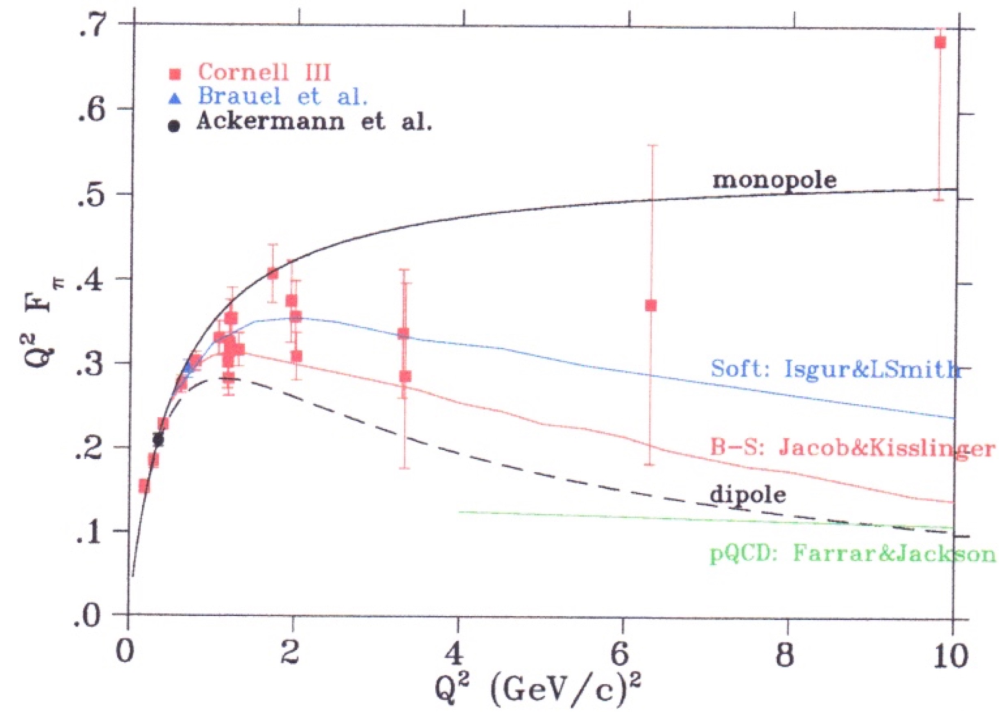




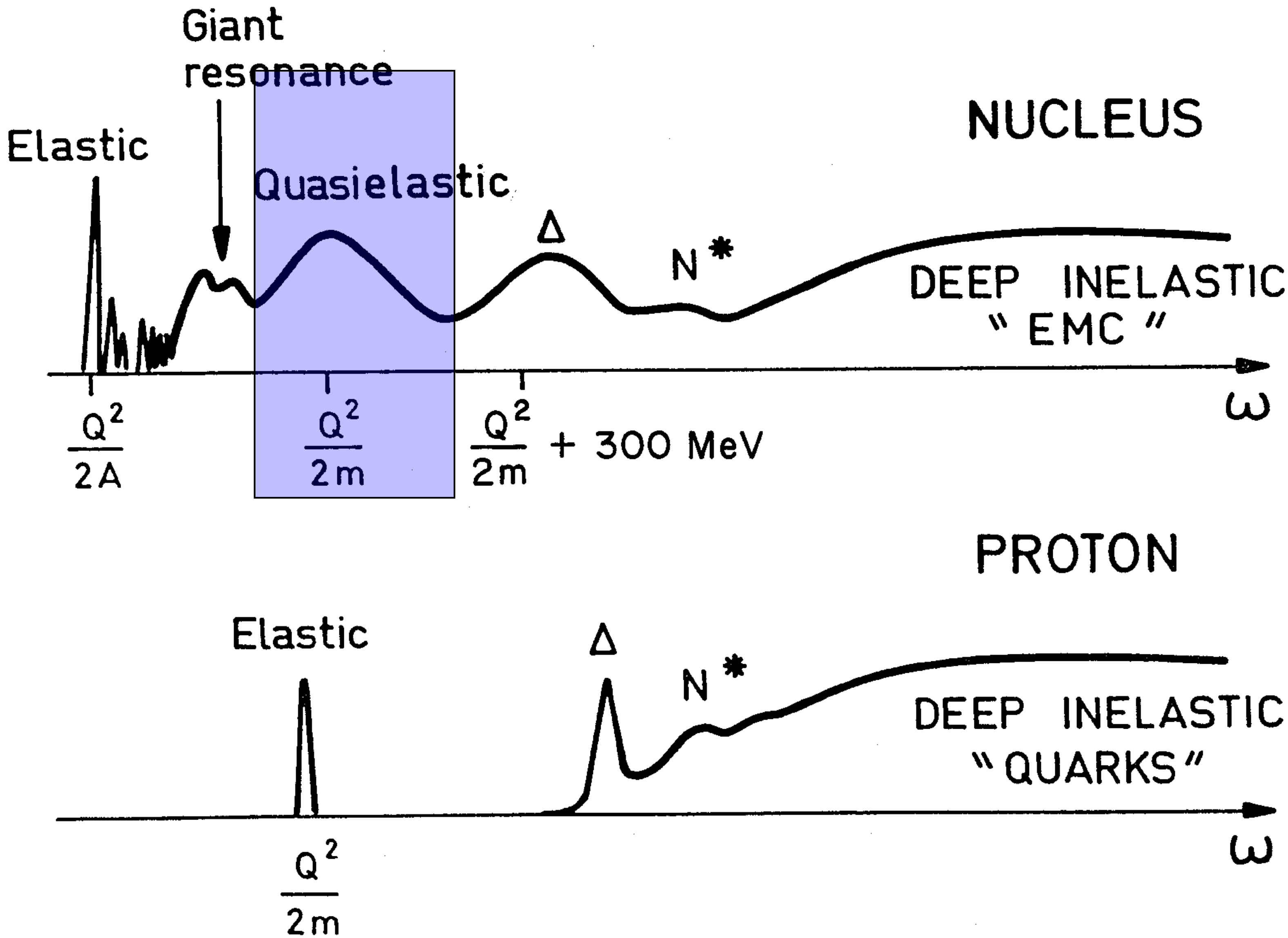
- Possible Explanation: 2-photon exchange

Elastic Form Factors of Hadrons

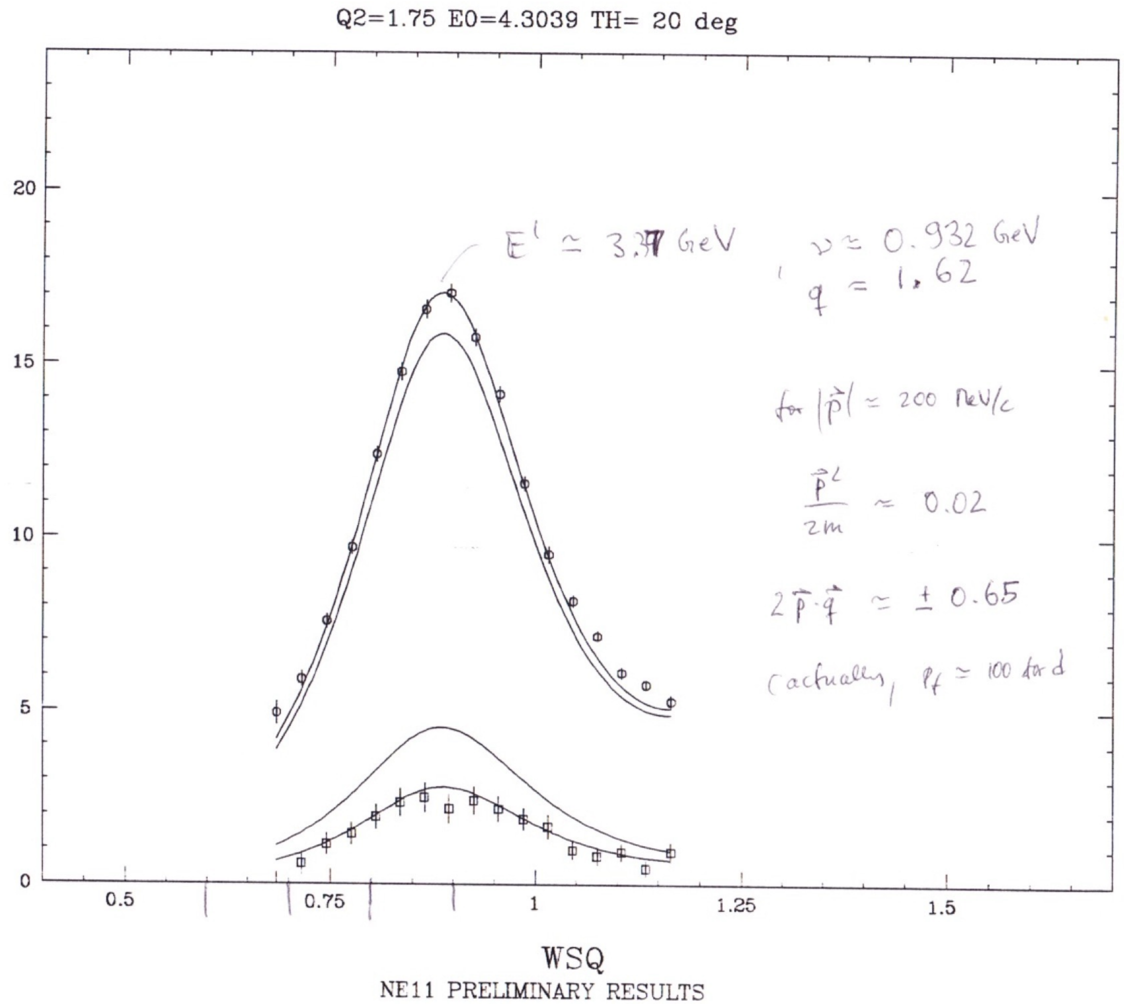
- Example: Pion



II. Quasielastic scattering



Quasi-elastic Scattering off Nuclei



-> scattering of single nucleons bound in the nucleus; momentum distribution, energy of residual system

Quasi-elastic Scattering off Nuclei with coincindent detection of proton

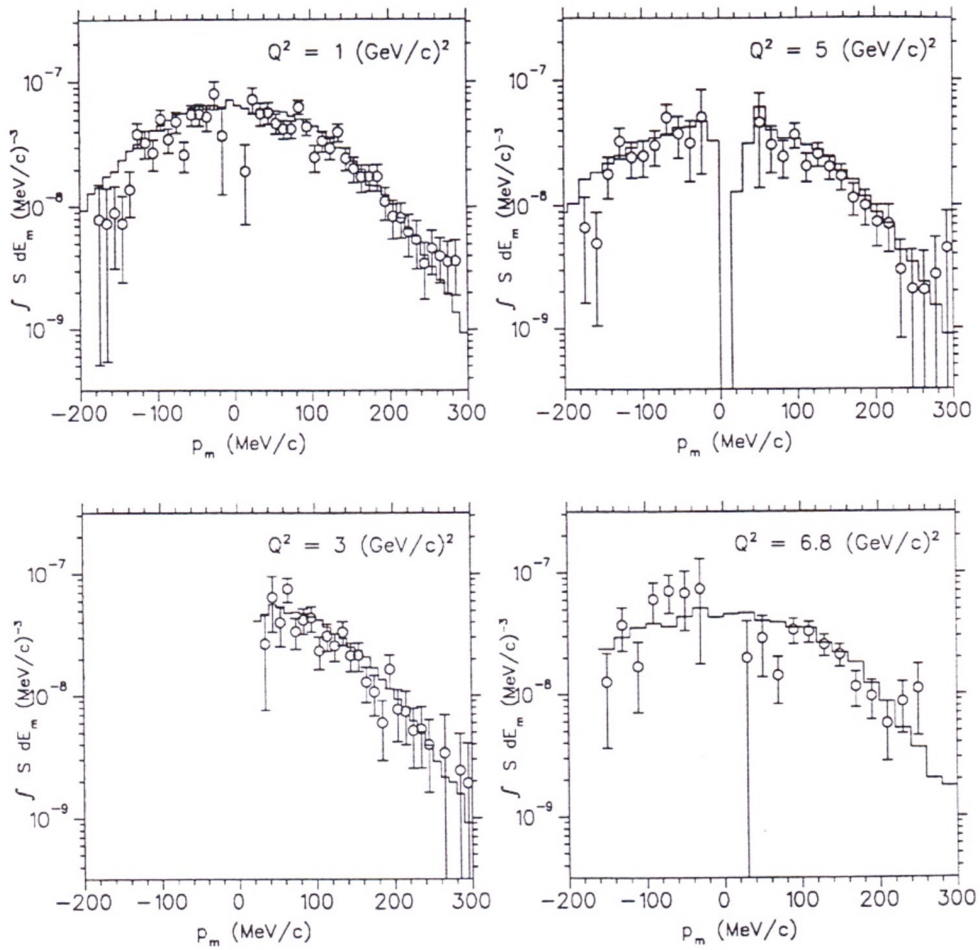
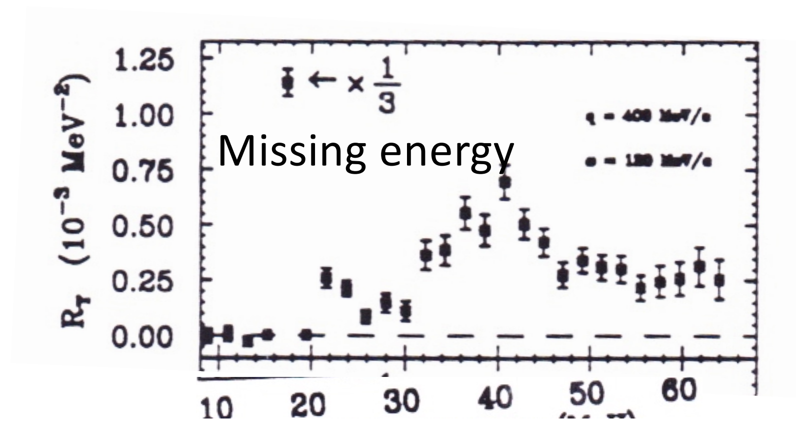


Figure 5-12: Extracted $\rho(p_m)$ for the 1s shell of ^{12}C. The spectral function has been integrated over $30 < E_m < 80$ MeV. The solid line represents the result of the PWIA calculation, normalized to the measured transparency; the data points are shown with statistical errors only.

-> momentum and energy distribution of bound proton; "Spectroscopic Factor"



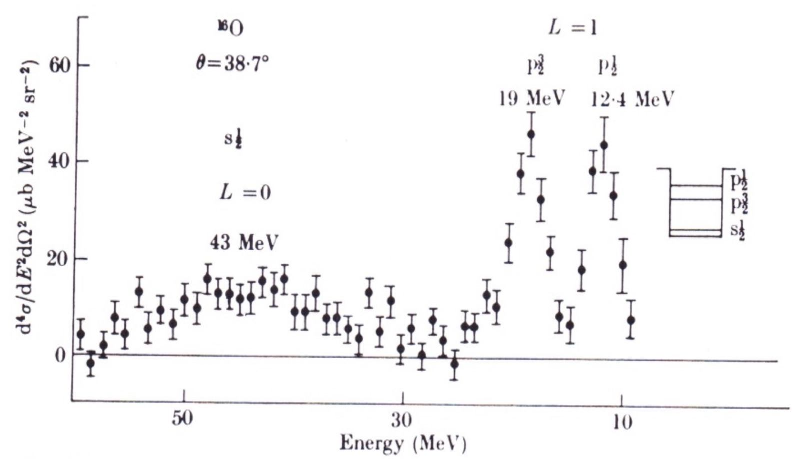
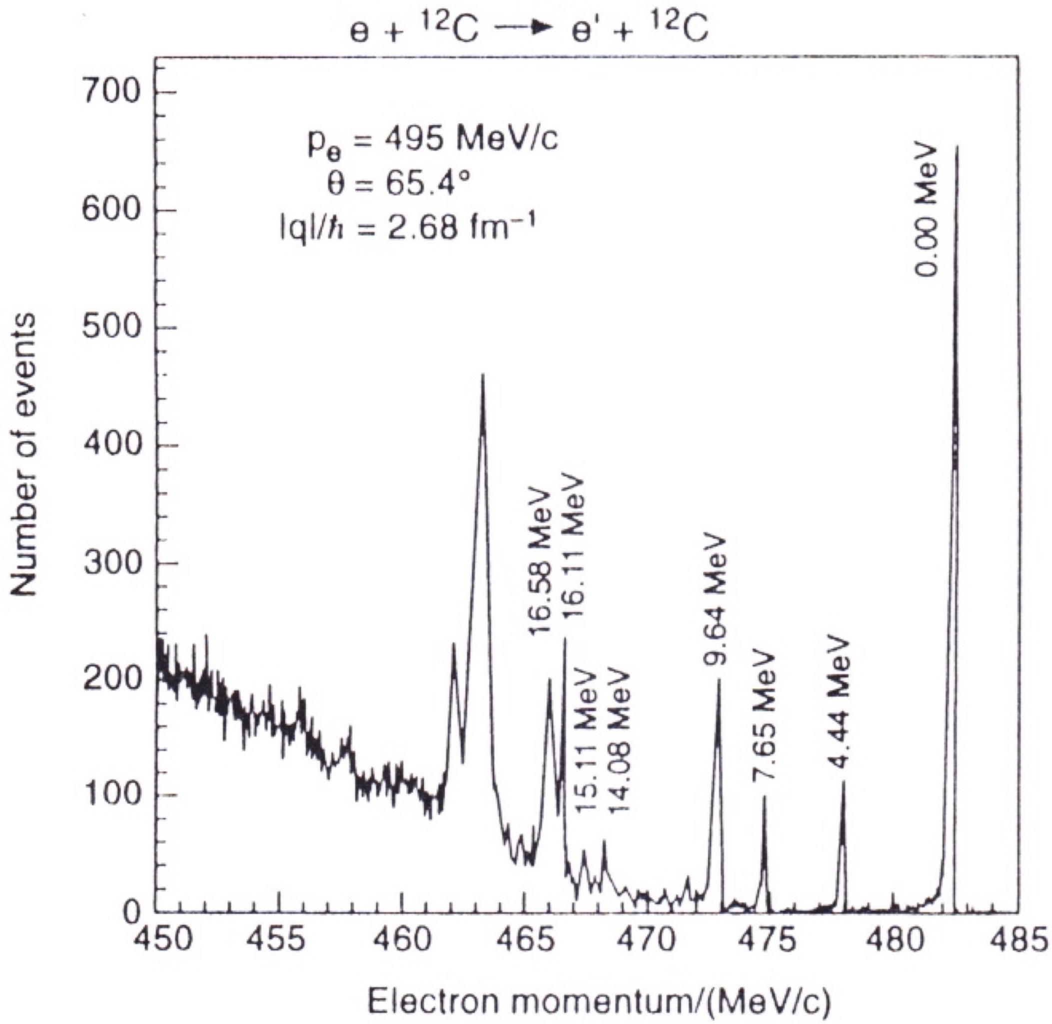


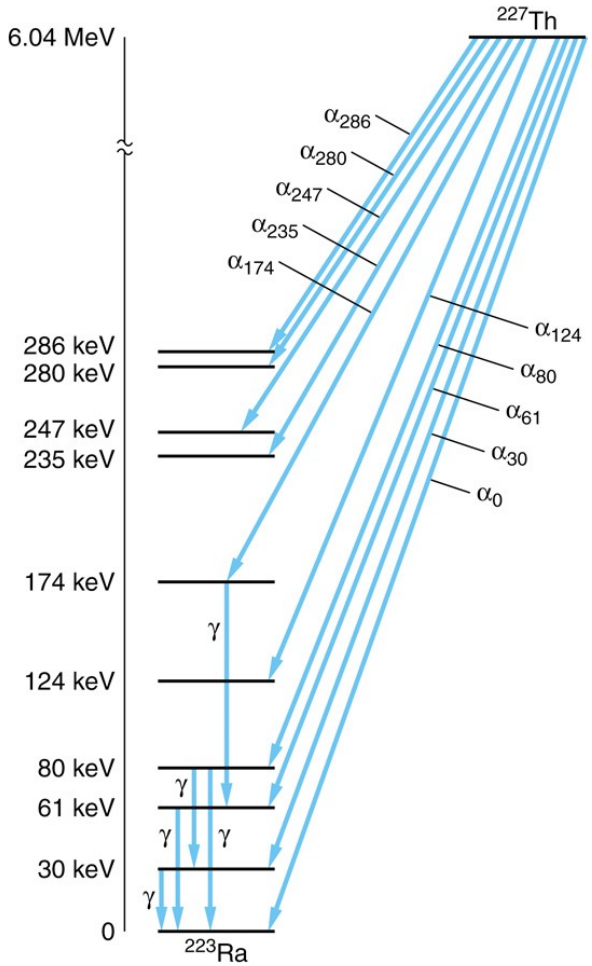
Fig. 17.8 Distribution of energies required to remove a proton from ^{16}O by the (p,2p) reaction, showing the single-particle states (Tyrén *et al.* 1958).

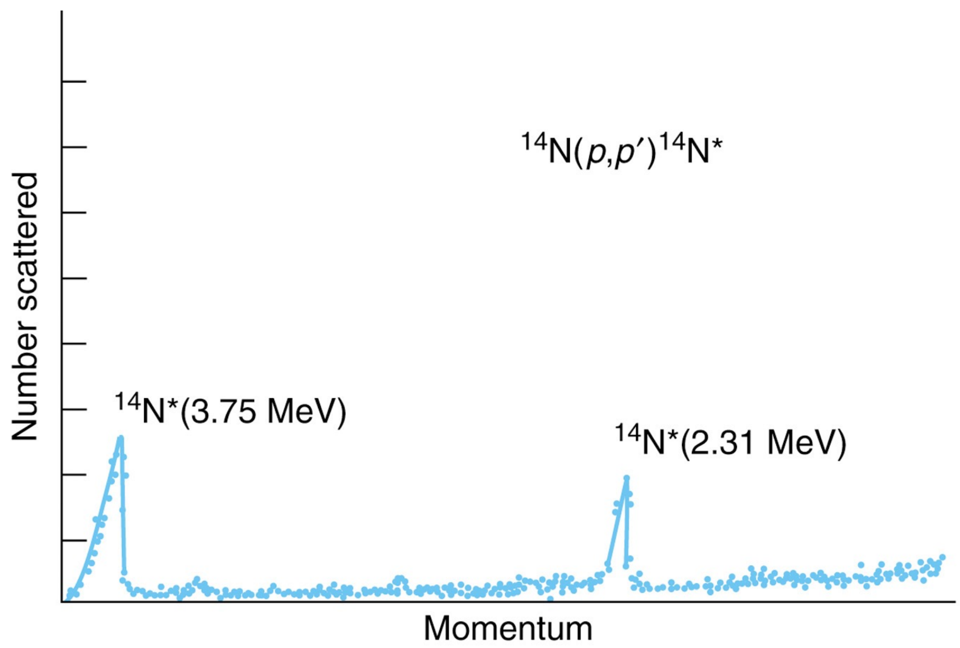
Inelastic Scattering off Nuclei



-> Excited states of nuclei, transition matrix elements

Evidence for excited states





Elastic scattering

$$\frac{\Delta\sigma}{\Delta\Omega} = \frac{4\alpha^2(\hbar c)^2 E'^2 cos^2\frac{\theta}{2}}{Q^4}\frac{E'}{E}\left(\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau\, tan^2\frac{\theta}{2}\, G_M^2(Q^2)\right)$$

where $\tau = \nu^2/Q^2$.

Inelastic Scattering

Note: $Q^2 = 2EE'(1-\cos\theta) \Rightarrow \Delta Q^2 = 2EE'\sin\theta\Delta\theta$, $\Delta\Omega = 2\pi \sin\theta\Delta\theta = \pi/EE'\ \Delta Q^2 \Rightarrow \Delta\sigma/\Delta Q^2 = \pi/EE'\ \Delta\sigma/\Delta\Omega$

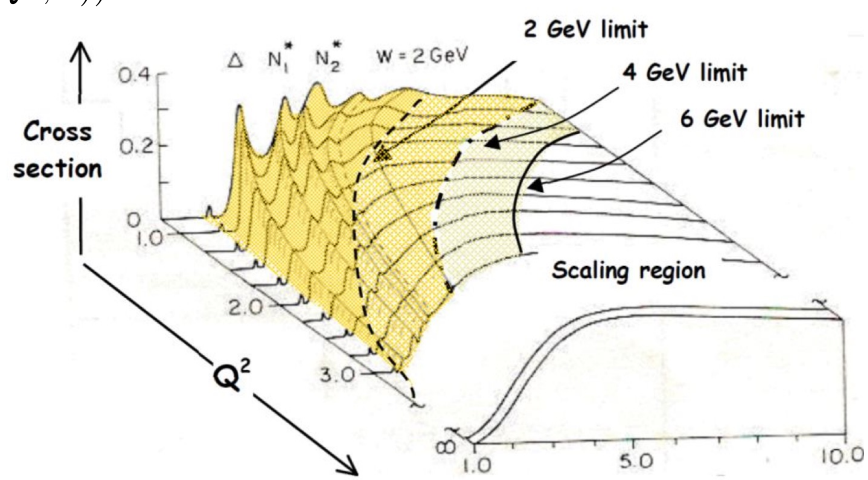
$x = \frac{p}{P}$

W: γ^*-p invariant mass

$$\frac{\Delta\sigma}{\Delta Q^2\Delta\nu} = \frac{4\pi\alpha^2(\hbar c)^2 E' cos^2(\theta/2)}{Q^4 E}(W_2(Q^2,\nu) + 2\,tan^2(\theta/2)\,W_1(Q^2,\nu))$$

$$\frac{\Delta\sigma}{\Delta Q^2\Delta\nu} = \frac{4\pi\alpha^2(\hbar c)^2 E' cos^2(\theta/2)}{Q^4 E}\frac{W_1(Q^2,\nu)}{\epsilon(1+\tau)}(1+\epsilon R(Q^2,\nu))$$

with $\epsilon = (1+2(1+\tau)tan^2(\theta/2))^{-1}$



$\omega' = \frac{1}{x} + \frac{m^2}{Q^2}$

$= W^2/Q^2 + 1$

$$\frac{\Delta\sigma}{\Delta Q^2\Delta\nu} = \frac{4\pi\alpha^2(\hbar c)^2 E' cos^2(\theta/2)}{Q^4 E}\left(\frac{1}{\nu}F_2(x) + 2\,tan^2(\theta/2)\frac{1}{M}F_1(x)\right)$$

$$F_1(x) = MW_1(Q^2,\nu) \qquad F_2(x) = \nu W_2(Q^2,\nu)$$

Only "mildly" dependent on Q^2

Picture from F. Gross,
«Making the case for Jefferson Lab»
The first decade of Science at Jefferson Lab
JoP, Conf. Series 299 (2011) 012001

Inelastic Cross Section – What's different

Phase space factor $d^3k' = k'^2 d\Omega_{k'} dk'$

(δ-function drops since E' can have any value)

— conversion —

$$k'^2 d\Omega dk' = E'^2 2\pi \sin\theta d\theta dE'$$
$$= -2E'^2 d\cos\theta \, \pi dE'$$
$$= \frac{\pi E'}{E} \underbrace{(-2EE' d\cos\theta)}_{+dQ^2} \underbrace{dE'}_{|d\nu|}$$

since $\nu = E - E'$

Replace $\frac{G_E^2 + \tau G_M^2}{1+\tau}$ with $W_2(Q^2, \underline{\underline{\nu}})$ *)

and τG_M^2 with $W_1(Q^2, \nu) \Rightarrow$

$$\Delta\sigma = \frac{4\pi\alpha^2(\hbar c)^2}{Q^4}\frac{E'}{E}\cos^2\theta/2 \left[W_2 + 2\tan^2\frac{\theta}{2} W_1\right] \Delta Q^2 \Delta\nu$$

$$*) \Rightarrow W_2^{el}(Q^2, \nu) = \frac{G_E^2 + \tau G_M^2}{1+\tau} \cdot \delta(\nu - \nu_{el})$$
$$W_1^{el}(Q^2, \nu) = \tau G_M^2 \cdot \delta(\nu - \nu_{el})$$

Elastic Cross section – final form

$$\Delta\sigma = \frac{4z^2\alpha^2(\hbar c)^2}{Q^4}\cos^2\frac{\theta}{2}\left[\underbrace{\frac{G_E^2 + \tau G_M^2}{1+\tau}}_{\text{LONGITUDINAL (charge)}} + \underbrace{2\tau\tan^2\frac{\theta}{2} G_M^2}_{\text{TRANSVERSE (magnetic)}}\right] \cdot E'^2 \Delta\Omega \underbrace{\frac{E'}{E}}_{\text{Recoil}}$$

$\tau = \frac{\nu^2}{Q^2}$, $G_E(Q^2)$, $G_M(Q^2)$: Form Factors

Dirac Particle : $G_E = G_M = 1$ (const.)

Anomalous magnetic moment : $G_M \approx (1+\kappa) G_E$

Extended Charge distribution :

$G_E(Q^2) \simeq$ Fourier transform of $\rho(r)$

Ex: $\rho(r) \simeq \frac{a^3}{8\pi} e^{-ar} \Rightarrow G_E(Q^2) \simeq \left(\frac{1}{1+\frac{Q^2}{a^2}}\right)^2$

(Dipole Form). $p: a^2 = 0.71 \text{ GeV}^2$

Inelastic Cross Section – What's different

Phase space factor $d^3k' = k'^2 d\Omega_{k'}\, dk'$

(δ-function drops since E' can have any value)

— conversion —

$$k'^2 d\Omega dk' = E'^2 2\pi \sin\theta\, d\theta\, dE'$$
$$= -2E'^2 d\cos\theta\, \pi\, dE'$$
$$= \frac{\pi E'}{E} \underbrace{(-2EE' d\cos\theta)}_{+dQ^2} \underbrace{dE'}_{|d\nu|} \quad \text{since } \nu = E - E'$$

Replace $\frac{G_E^2 + \tau G_M^2}{1+\tau}$ with $W_2(Q^2, \underline{\underline{\nu}})$ *)

and τG_m^2 with $W_1(Q^2, \nu) \Rightarrow$

$$\Delta\sigma = \frac{4\pi\alpha^2(\hbar c)^2}{Q^4} \frac{E'}{E} \cos^2\theta/2 \left[W_2 + 2\tan^2\frac{\theta}{2} W_1 \right] \Delta Q^2 \Delta\nu$$

*) $\Rightarrow$ $W_2^{el}(Q^2, \nu) = \frac{G_E^2 + \tau G_M^2}{1+\tau} \cdot \delta(\nu - \nu_{el})$

$W_1^{el}(Q^2, \nu) = \tau G_m^2 \cdot \delta(\nu - \nu_{el})$

Interpretation of $W_1(Q^2, \nu)$

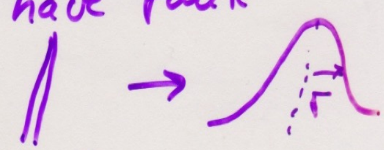
(transverse) electromagnetic transition probability to final state characterized by ν, with resolution $\sim \frac{1}{\sqrt{Q^2}}$

Transition to discrete final states (resonances):

$$W_1(Q^2, \nu) \simeq \tau G_m^2(\text{transition}) \cdot \delta(\nu - \nu_R)$$

ν_R is given by $M_{Res}^2 \stackrel{!}{=} W_{Res}^2 = M^2 + 2M\nu_R - Q^2$

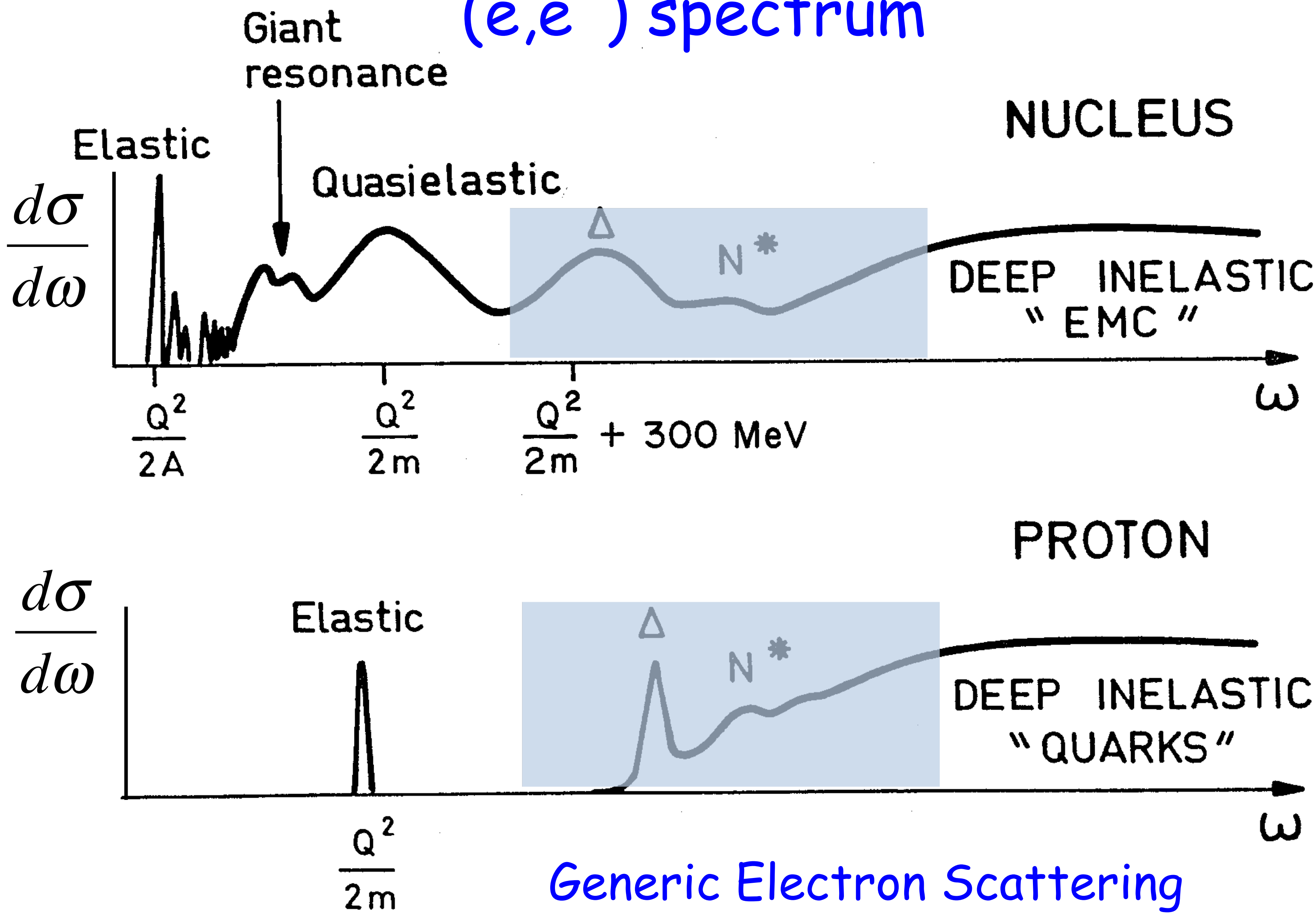
In reality: Resonances have finite width $\Gamma \sim \frac{1}{\tau_{Life}} \Rightarrow$



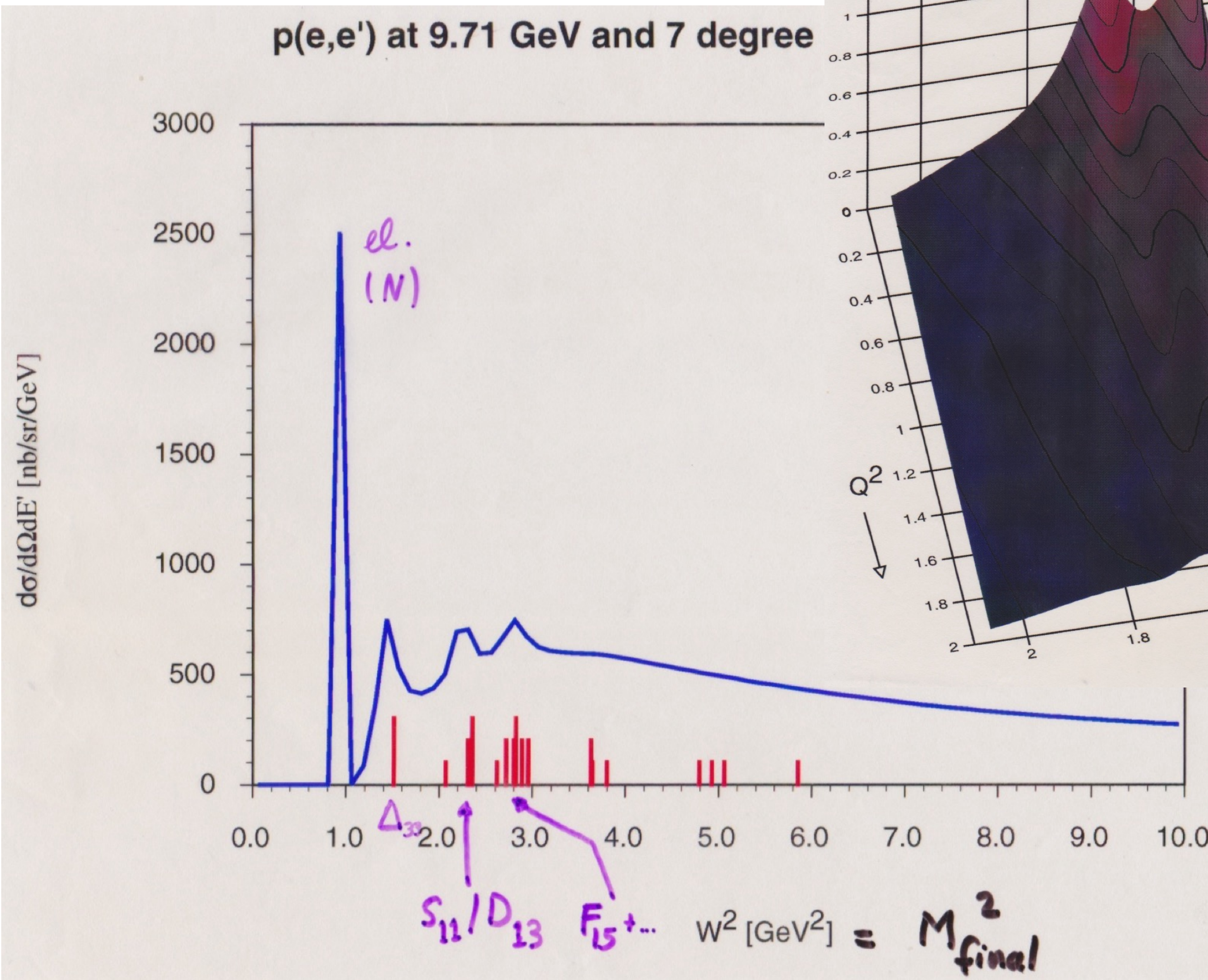
(except elastic transition) δ-function

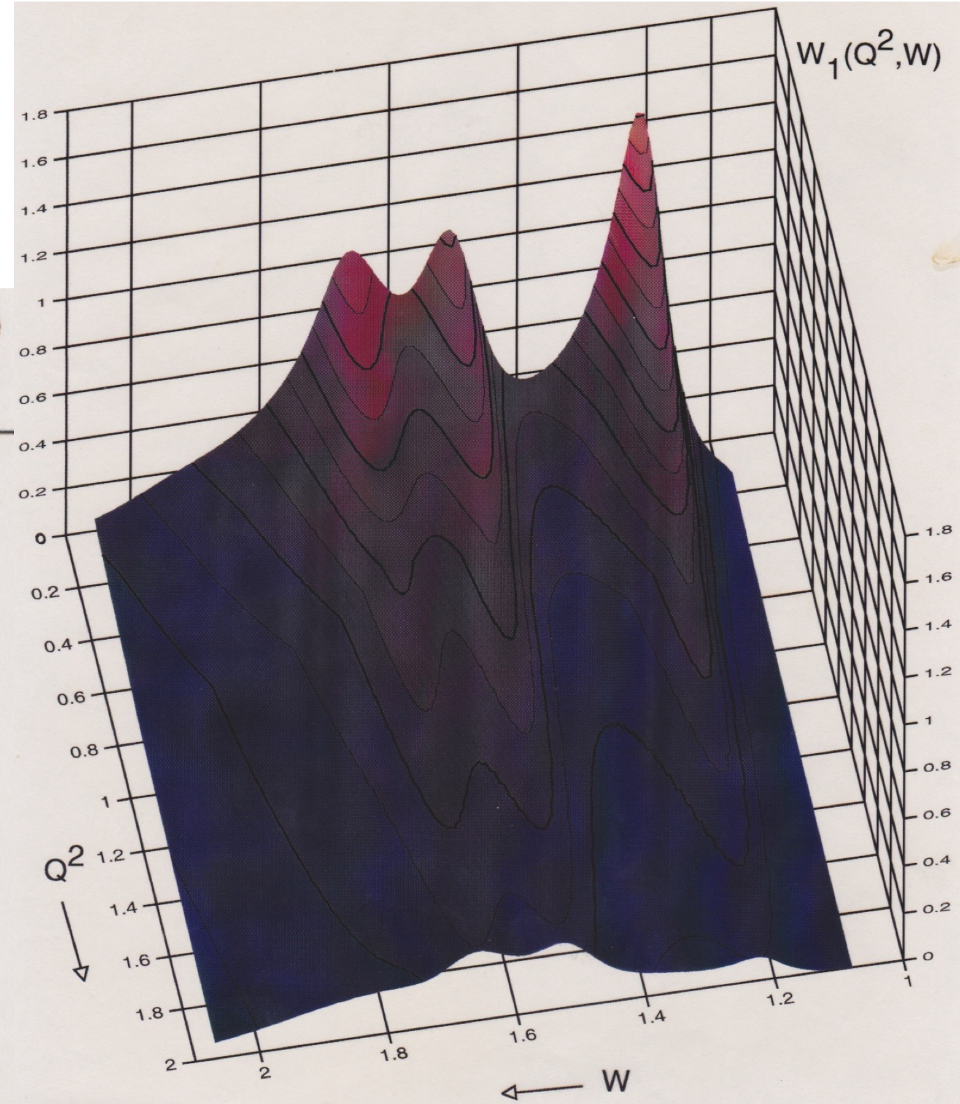
Threshold: $W_{min}^2 = (M + m_\pi)^2 = (1.08\text{ GeV})^2$

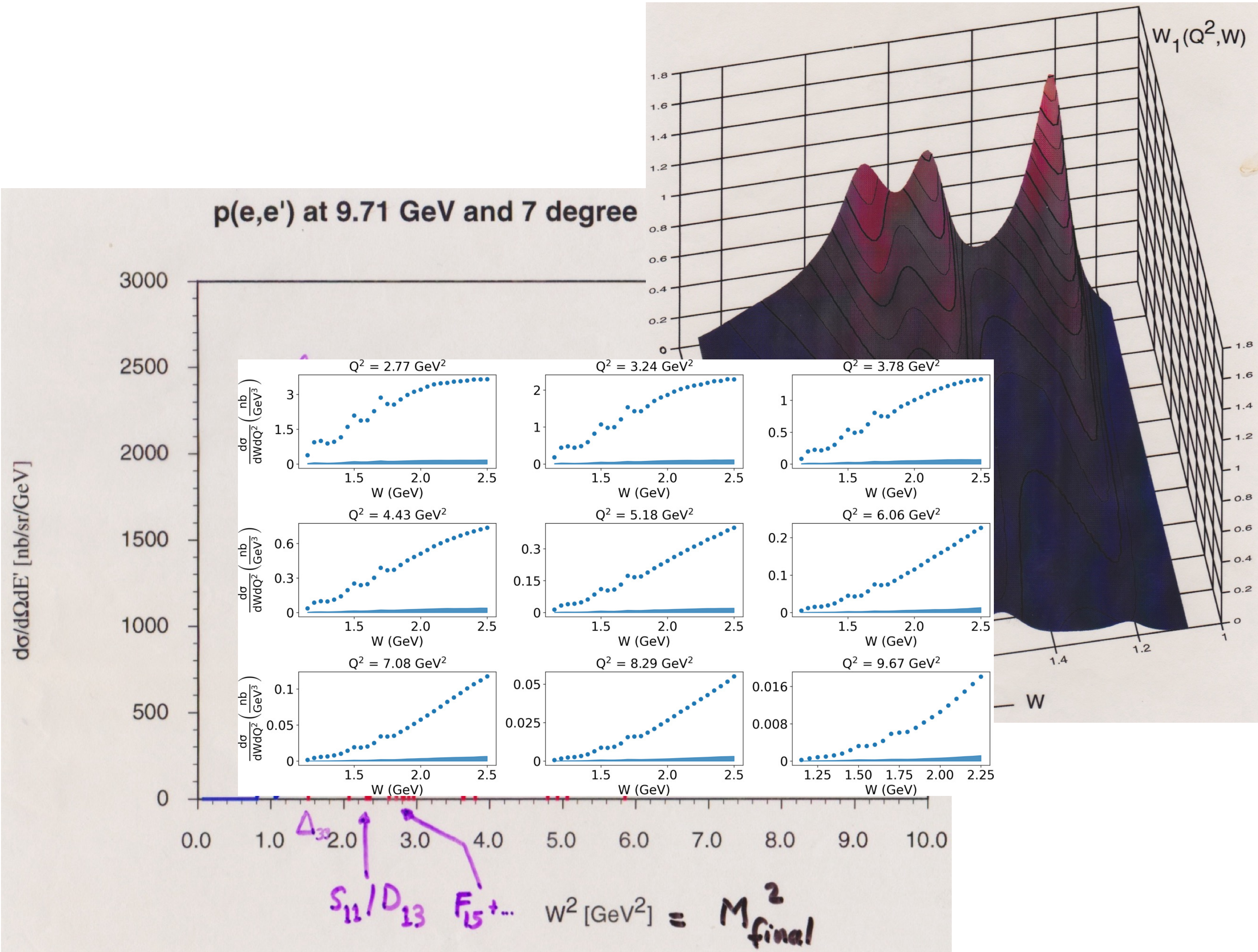
(e,e') spectrum



Generic Electron Scattering
at fixed momentum transfer


p(e,e') at 9.71 GeV and 7 degree
dσ/dΩdE' [nb/sr/GeV]
3000
2500
2000
1500
1000
500
0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
el.
(N)
Δ_{33}
S_{11} / D_{13}
F_{15} +...
W² [GeV²] = M^2_{final}


$W_1(Q^2,W)$
Q^2
W



p(e,e') at 9.71 GeV and 7 degree
dσ/dΩdE' [nb/sr/GeV]
3000
2500
2000
1500
1000
500
0
0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
Δ33
S11/D13
F15+...
W2 [GeV2] = M2 final
W1(Q2,W)
W
Q2 = 2.77 GeV2
Q2 = 3.24 GeV2
Q2 = 3.78 GeV2
Q2 = 4.43 GeV2
Q2 = 5.18 GeV2
Q2 = 6.06 GeV2
Q2 = 7.08 GeV2
Q2 = 8.29 GeV2
Q2 = 9.67 GeV2
dσ/dWdQ2 (nb/GeV3)
W (GeV)