QCD Made Simple

Frank Wilczek 22 August 2000 Physics Today

$$\mathcal{L} = \frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \sum_j \bar{q}_j (i\gamma^\mu D_\mu + m_j) q_j$$

$$\text{where } G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^a_{bc} A^b_\mu A^c_\nu$$

$$\text{and } D_\mu \equiv \partial_\mu + i t^a A^a_\mu$$

<u>That's it!</u>

Figure 1. The QCD Lagrangian $\mathcal{L}$ displayed here is, in principle, a complete description of the strong interaction. But, in practice, it leads to equations that are notoriously hard to solve. Here m_j and q_j are the mass and quantum field of the quark of jth flavor, and A is the gluon field, with spacetime indices μ and ν and color indices a, b, c. The numerical coefficients f and t guarantee SU(3) color symmetry. Aside from the quark masses, the one coupling constant g is the only free parameter of the theory.

How many quarks? From PDG

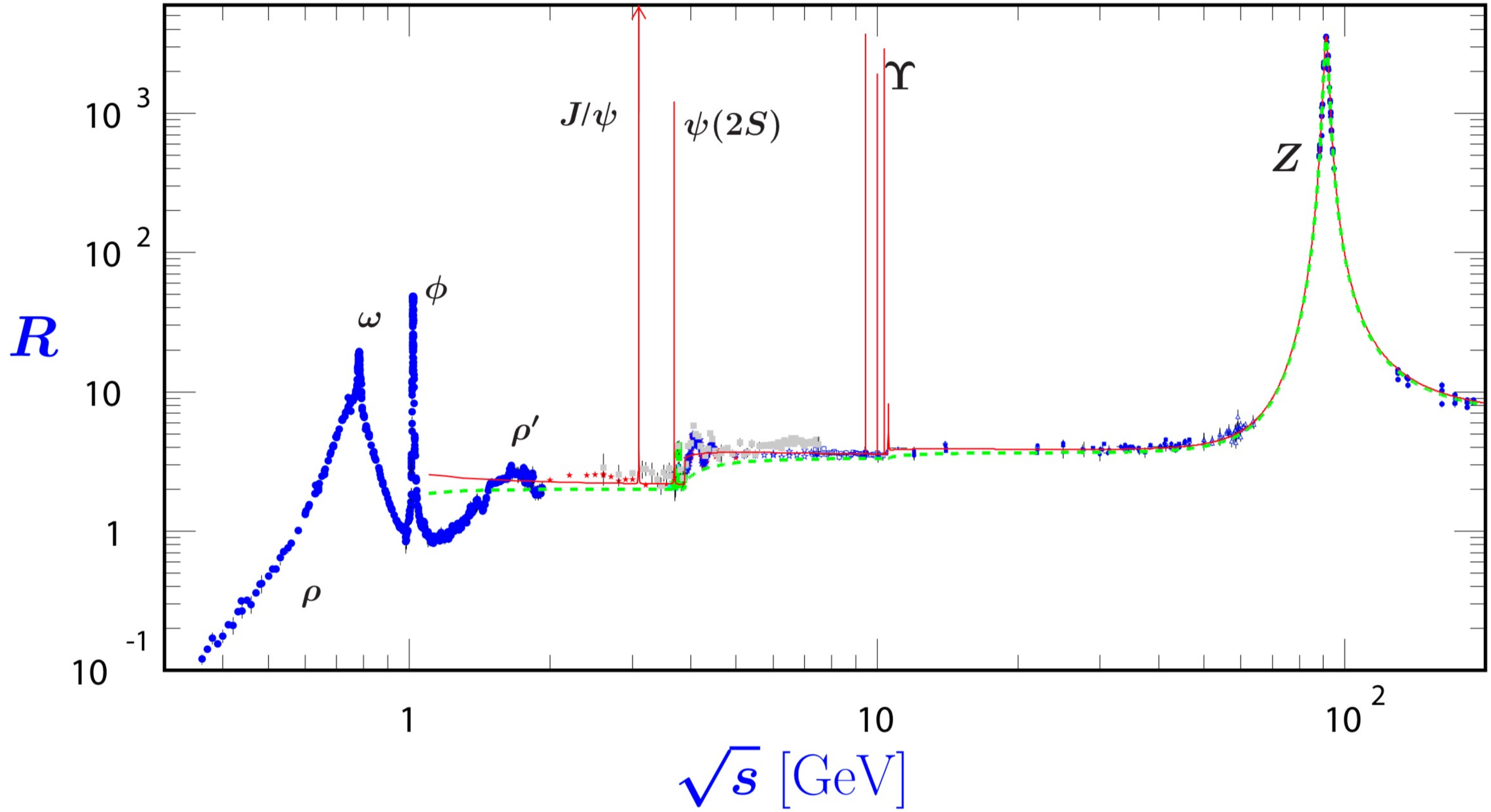
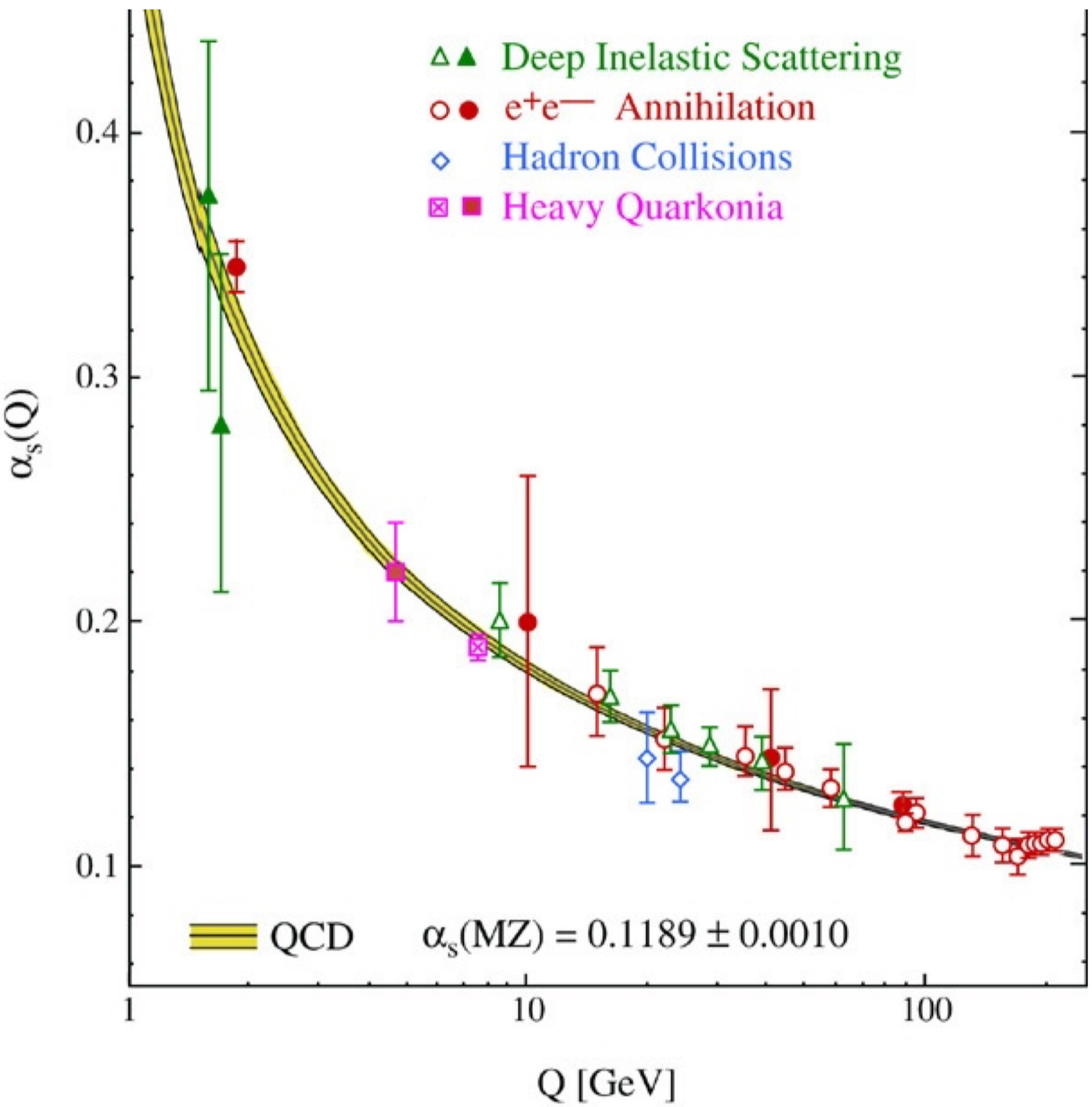


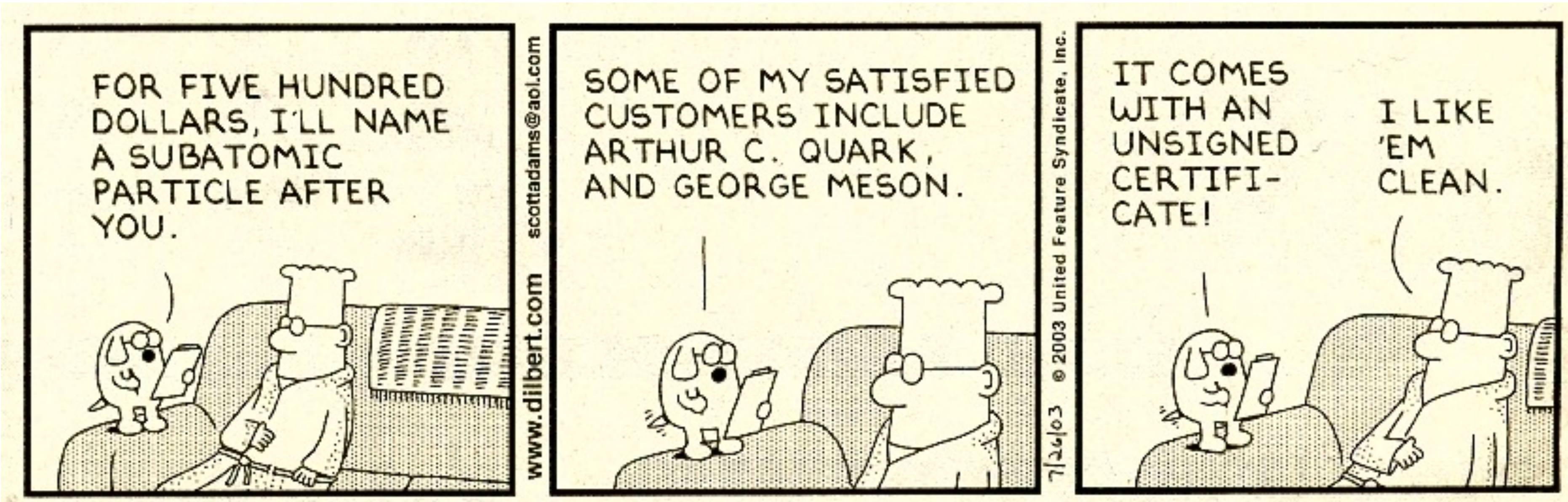
Figure 51.2: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. Corresponding computer-readable data files are available at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, August 2017. Corrections by P. Janot (CERN) and M. Schmitt (Northwestern U.))

Running of the Strong Coupling Constant



$$\alpha_S(\mu) = \frac{12\pi}{(33 - 2n_f)\ln(\mu^2 / \Lambda^2)}$$

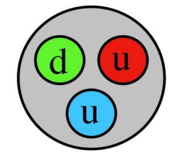
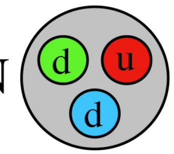
$$\Lambda_{QCD} \approx 0.25 \text{ GeV}$$



See also **http://particleadventure.org**

Hadronic Particle Zoo

- what can one build from quarks?

P  N 

HADRONS

Family Name	Particle Name	Particle Symbol	Antiparticle Symbol	Composition	Mass	Electric Charge	Lifetime in Seconds
baryon	proton	p or p^+	$\bar{p}$	uud	1,836	+1	stable
	neutron	n or n^0	$\bar{n}$	udd	1,839	0	887
	lambda	Λ^0	$\bar{\Lambda}^0$	uds	2,183	0	2.6×10^{-10}
	lambda-c	Λ^+_c	$\bar{\Lambda}^-_c$	udc	4,471	+1	2.1×10^{-13}
	lambda-b	Λ^0_b	$\bar{\Lambda}^0_b$	udb	11,000	0	1.1×10^{-12}
	sigma	Σ^+	$\bar{\Sigma}^+$	uus	2,328	+1	0.8×10^{-10}
		Σ^0	$\bar{\Sigma}^0$	$\frac{(ud \pm du)s}{\sqrt{2}}$	2,334	0	7.4×10^{-20}
		Σ^-	$\bar{\Sigma}^-$	dds	2,343	−1	1.5×10^{-10}
	xi	Ξ^0	$\bar{\Xi}^0$	uss	2,573	0	2.9×10^{-10}
		Ξ^-	$\bar{\Xi}^+$	dss	2,585	−1	1.6×10^{-10}
	xi-c	Ξ^0_c	$\bar{\Xi}^0_c$	dsc	4,834	0	9.8×10^{-14}
		Ξ^+_c	$\bar{\Xi}^-_c$	usc	4,826	+1	3.5×10^{-13}
	omega	Ω^-	Ω^+	sss	3,272	−1	0.8×10^{-10}
	omega-c	Ω^0_c	$\bar{\Omega}^0_c$	ssc	5,292	0	6.4×10^{-14}
meson	pion	π^+	π^-	$u\bar{d}$	273	+1	2.6×10^{-8}
		π^0	π^0	$\frac{(u\bar{u} - d\bar{d})}{\sqrt{2}}$	264	0	8.4×10^{-17}
	kaon*	K^+	K^-	$u\bar{s}$	966	+1	1.2×10^{-8}
		K^0	$\bar{K}^0$	$d\bar{s}$	974	0	8.9×10^{-11} 5.2×10^{-8}
	J/psi	J or Ψ	J or Ψ	$c\bar{c}$	6,060	0	1.0×10^{-20}
	omega	ω	ω	$\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}}$	1,532	0	6.6×10^{-23}
	eta	η	η	$\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}}$	1,071	0	3.5×10^{-22}
	eta-c	η_c	η_c	$c\bar{c}$	5,832	0	3.1×10^{-22}
	B	B^0	$\bar{B}^0$	$d\bar{b}$	10,331	0	1.6×10^{-12}
		B^+	B^-	$u\bar{b}$	10,331	+1	1.6×10^{-12}
	B-s	B^0_s	$\bar{B}^0_s$	$s\bar{b}$	10,507	0	1.6×10^{-12}
	D	D_0	$\bar{D}_0$	$c\bar{u}$	3,649	0	4.2×10^{-13}
		D^+	D^-	$c\bar{d}$	3,658	+1	1.1×10^{-12}
	D-s	D^+_s	D^-_s	$c\bar{s}$	3,852	+1	4.7×10^{-13}
	chi	χ^0_c	χ^0_c	$c\bar{c}$	6,687	0	3.0×10^{-22}
	psi	Ψ^0_c	Ψ^0_c	$c\bar{c}$	7,213	0	1.5×10^{-20}
	upsilon	Υ	Υ	$b\bar{b}$	18,513	0	8.0×10^{-20}

*The neutral kaon is composed of two particles; the average lifetime of each particle is given.

Angular Momentum, Spin, Parity, …

- Total angular momentum J is sum of orbital angular momentum L and Spin S
- Parity P = + or - = behavior under reflections
  - Antiparticles have opposite parity to particles
  - Angular momentum eigenstates with L = 0,1,2… have parity +,-,+,…
- …and Charge Symmetry C

$\pi^{\pm}$

$I^G(J^P) = 1^-(0^-)$

Mass $m = 139.57039 \pm 0.00018$ MeV (S = 1.8)
Mean life $\tau = (2.6033 \pm 0.0005) \times 10^{-8}$ s (S = 1.2)
$c\tau = 7.8045$ m

$\pi^{\pm} \rightarrow \ell^{\pm}\nu\gamma$ form factors [a]

$F_V = 0.0254 \pm 0.0017$
$F_A = 0.0119 \pm 0.0001$
F_V slope parameter $a = 0.10 \pm 0.06$
$R = 0.059^{+0.009}_{-0.008}$

π^- modes are charge conjugates of the modes below.

For decay limits to particles which are not established, see the section on Searches for Axions and Other Very Light Bosons.

π^+ DECAY MODES		Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\mu^+\nu_\mu$	[b]	(99.98770 ± 0.00004) %		30

π^0

$I^G(J^{PC}) = 1^-(0^{-+})$

Mass $m = 134.9768 \pm 0.0005$ MeV (S = 1.1)
$m_{\pi^{\pm}} - m_{\pi^0} = 4.5936 \pm 0.0005$ MeV
Mean life $\tau = (8.43 \pm 0.13) \times 10^{-17}$ s (S = 1.2)
$c\tau = 25.3$ nm

For decay limits to particles which are not established, see the appropriate Search sections (A^0 (axion) and Other Light Boson (X^0) Searches, etc.).

π^0 DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
2γ	(98.823 ± 0.034) %	S=1.5	67
$e^+e^-\gamma$	$(\ 1.174 \pm 0.035)$ %	S=1.5	67

$K^{\pm}$

$I(J^P) = \frac{1}{2}(0^-)$

Mass $m = 493.677 \pm 0.015$ MeV [o] (S = 2.8)
Mean life $\tau = (1.2380 \pm 0.0020) \times 10^{-8}$ s (S = 1.8)
$c\tau = 3.711$ m

K^0

$I(J^P) = \frac{1}{2}(0^-)$

50% K_S, 50% K_L

Mass $m = 497.611 \pm 0.013$ MeV (S = 1.2)
$m_{K^0} - m_{K^{\pm}} = 3.934 \pm 0.020$ MeV (S = 1.6)

$D^{\pm}$

$I(J^P) = \frac{1}{2}(0^-)$

Mass $m = 1869.66 \pm 0.05$ MeV
Mean life $\tau = (1033 \pm 5) \times 10^{-15}$ s
$c\tau = 309.8$ μm

η

$I^G(J^{PC}) = 0^+(0^{-+})$

Mass $m = 547.862 \pm 0.017$ MeV
Full width $\Gamma = 1.31 \pm 0.05$ keV

(… and B-mesons…)

$\rho(770)$

$I^G(J^{PC}) = 1^+(1^{--})$

See the review on "Spectroscopy of Light Meson Resonances."

T-Matrix Pole $\sqrt{s} = (761\text{–}765) - i\,(71\text{–}74)$ MeV

ρ^0 mass (Breit-Wigner) $= 775.26 \pm 0.23$ MeV [g]

$\rho^\pm$ mass (Breit-Wigner) $= 775.11 \pm 0.34$ MeV [h]

ρ^0 full width (Breit-Wigner) $= 147.4 \pm 0.8$ MeV [g] (S = 2.0)

$\rho^\pm$ full width (Breit-Wigner) $= 149.1 \pm 0.8$ MeV [h]

$\rho(770)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
$\pi\pi$	$\sim$ 100 %		363

$\phi(1020)$

$I^G(J^{PC}) = 0^-(1^{--})$

Mass $m = 1019.460 \pm 0.016$ MeV

Full width $\Gamma = 4.249 \pm 0.013$ MeV (S = 1.1)

$\phi(1020)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
K^+K^-	(49.9 ± 0.5) %	S=1.5	127
$K^0_L K^0_S$	(33.6 ± 0.4) %	S=1.3	110
$\rho\pi + \pi^+\pi^-\pi^0$	(14.9 ± 0.4) %	S=1.3	–

$J/\psi(1S)$

$I^G(J^{PC}) = 0^-(1^{--})$

Mass $m = 3096.900 \pm 0.006$ MeV

Full width $\Gamma = 92.6 \pm 1.7$ keV (S = 1.1)

(…+bottonium…)

…and all L>0 states…

(208 pages in PDG)

BARYONS:
Simple (Constituent) Quark Model

Flavor	Isospin I	I_3	Strangeness S	Charge Q	Baryon Number B
U	$1/2$	$+1/2$	0	$+2/3$	$1/3$
D	$1/2$	$-1/2$	0	$-1/3$	$1/3$
S	0	0	-1	$-1/3$	$1/3$

$$|\Delta^{++}\uparrow\rangle = |U\uparrow U\uparrow U\uparrow\rangle$$

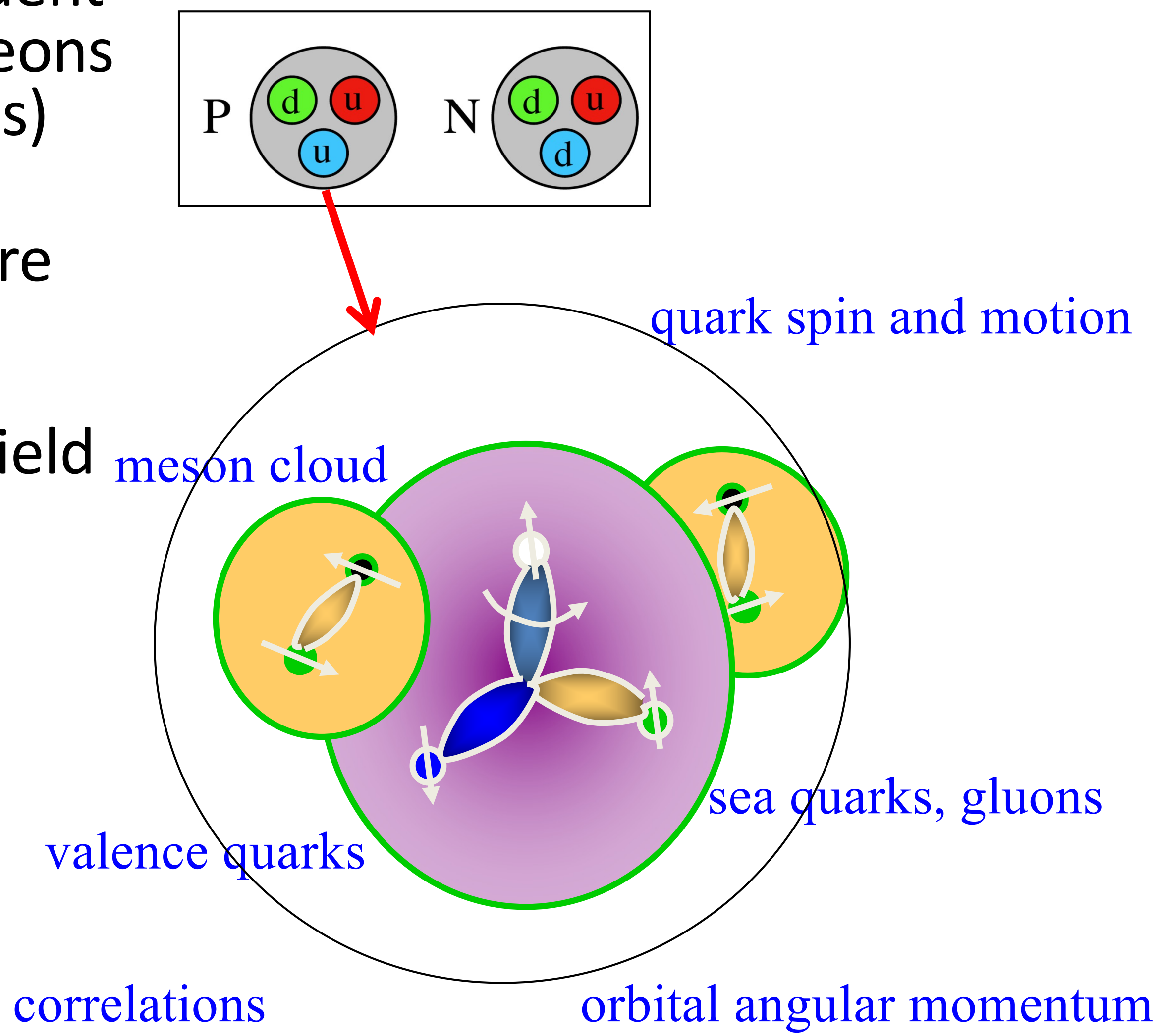
$$|\Delta^{+}\uparrow\rangle = 1/\sqrt{3}\,(|U\uparrow U\uparrow D\uparrow\rangle + |U\uparrow D\uparrow U\uparrow\rangle + |D\uparrow U\uparrow U\uparrow\rangle)$$

The case of the proton is a bit more complicated, since the wave function cannot be symmetric in spin and flavor separately. The most intuitive way to derive the proton wave function is by observing that 2 of the 3 quarks are equal (U), and therefore their relative spin wave function should be symmetric also. This leads to the conclusion that the two U–quarks couple their spins to a total spin of one. Let's denote the case where this spin has a z-projection of $+1$ as $(UU\Uparrow) := |U\uparrow U\uparrow\rangle$, while the projection with $S_z = 0$ will be indicated by $(UU\Rightarrow) := 1/\sqrt{2}\,(|U\uparrow U\downarrow\rangle + |U\downarrow U\uparrow\rangle)$. We can now combine the spin $1/2$ of the remaining D quark with the spin 1 of the UU pair in two ways to get total spin and projection $1/2$; the proper way follows simply from insertion of the correct Clebsch-Gordon coefficients:

$$|P\uparrow\rangle = 1/\sqrt{3}\left(\sqrt{2}|(UU\Uparrow)D\downarrow\rangle - |(UU\Rightarrow)D\uparrow\rangle\right). \qquad (2)$$

Hadron Structure

- Simple-most (constituent quark) model of nucleons (protons and neutrons)
- … becomes much more complicated once we consider the full relativistic quantum field theory called QCD
- Effective theories: Quark model, χPT, sum rules, …
- and Lattice QCD!

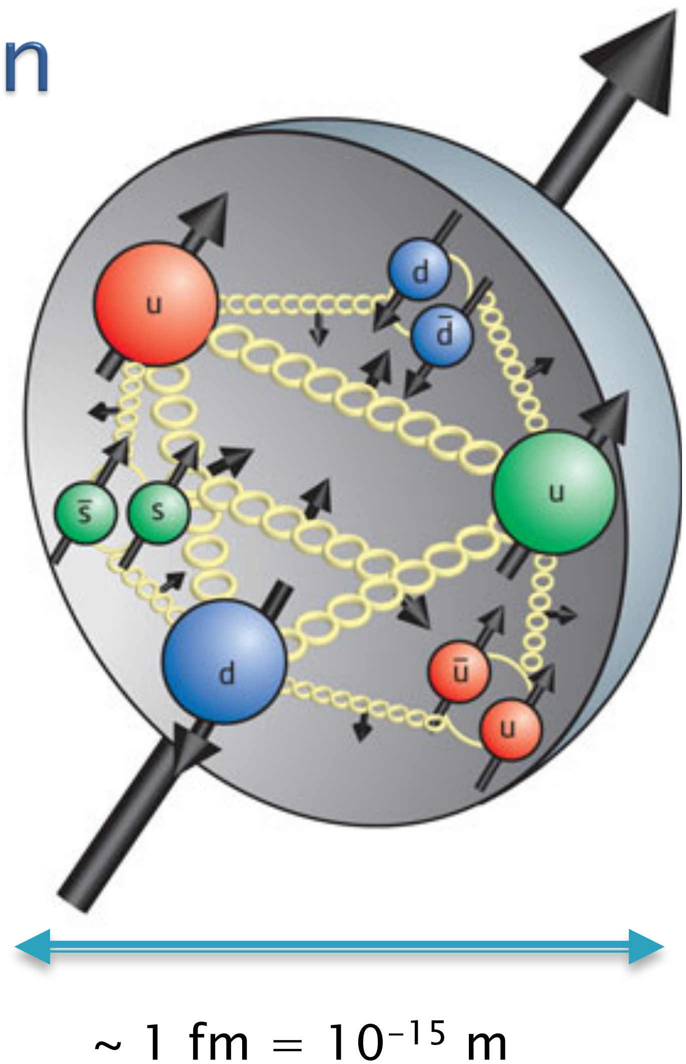


Reminder:

The structure of the nucleon

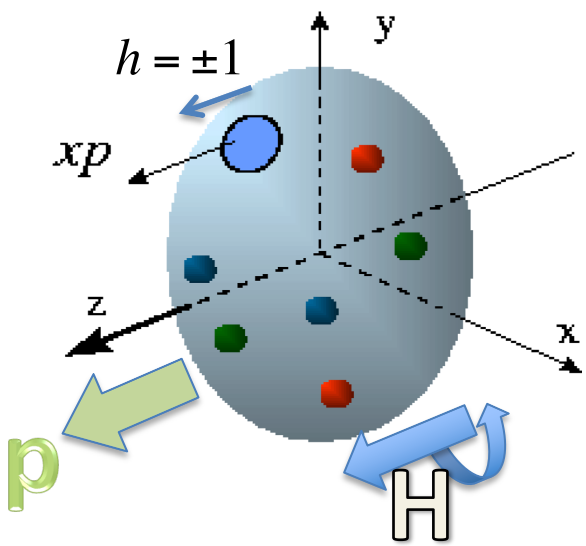
Three ”valence” quarks plus gluons plus quark/anti-quark pairs.

- How do the quarks and gluons interact to form a proton or neutron?
- Where does the spin of the nucleon come from?
- Why is the radius of the proton different when measured by electrons and muons?
- Why are protons and neutrons modified in a nucleus?



Inclusive lepton scattering

Parton model: DIS can access



$$q(x;Q^2),\langle h\cdot H\rangle q(x;Q^2)$$

Traditional "1-D" Parton Distributions (PDFs)
(integrated over many variables)

Callan-Gross

Wandzura-Wilczek

$$F_1(x)=\tfrac{1}{2}\sum_i e_i^2 q_i(x)\ \left(\text{and}\ \boxed{F_2(x)\approx 2xF_1(x)}\right)$$

$$g_1(x)=\tfrac{1}{2}\sum_i e_i^2 \Delta q_i(x)\ \left(\text{and}\ g_2(x)\approx \boxed{-g_1(x)+\int_x^1 \frac{g_1(y)}{y}dy}\right)$$

At finite Q^2: pQCD evolution ($q(x,Q^2)$, $\Delta q(x,Q^2)$ $\Rightarrow$ DGLAP equations), and gluon radiation

$$g_1(x,Q^2)_{pQCD}=\frac{1}{2}\sum_q^{N_f} e_q^2[(\Delta q+\Delta\bar{q})\otimes(1+\frac{\alpha_s(Q^2)}{2\pi}\delta C_q)+\frac{\alpha_s(Q^2)}{2\pi}\Delta G\otimes\frac{\delta C_G}{N_f}]$$

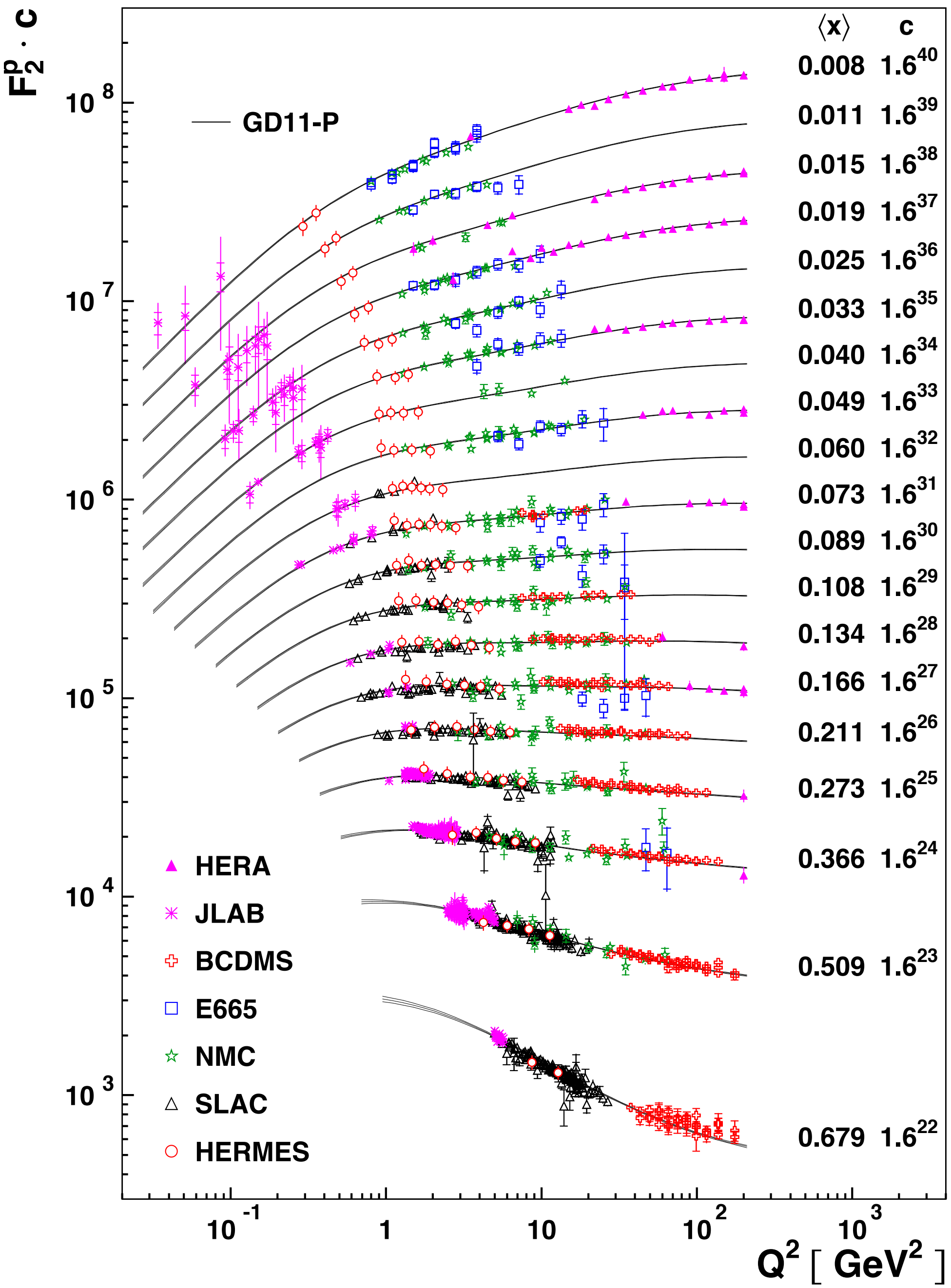
$\Rightarrow$ access to gluons. $\delta C_q, \delta C_G$ – *Wilson* coefficient functions

SIDIS: Tag the flavor of the struck quark with the leading FS hadron $\Rightarrow$ separate $q_i(x,Q^2)$, $\Delta q_i(x,Q^2)$

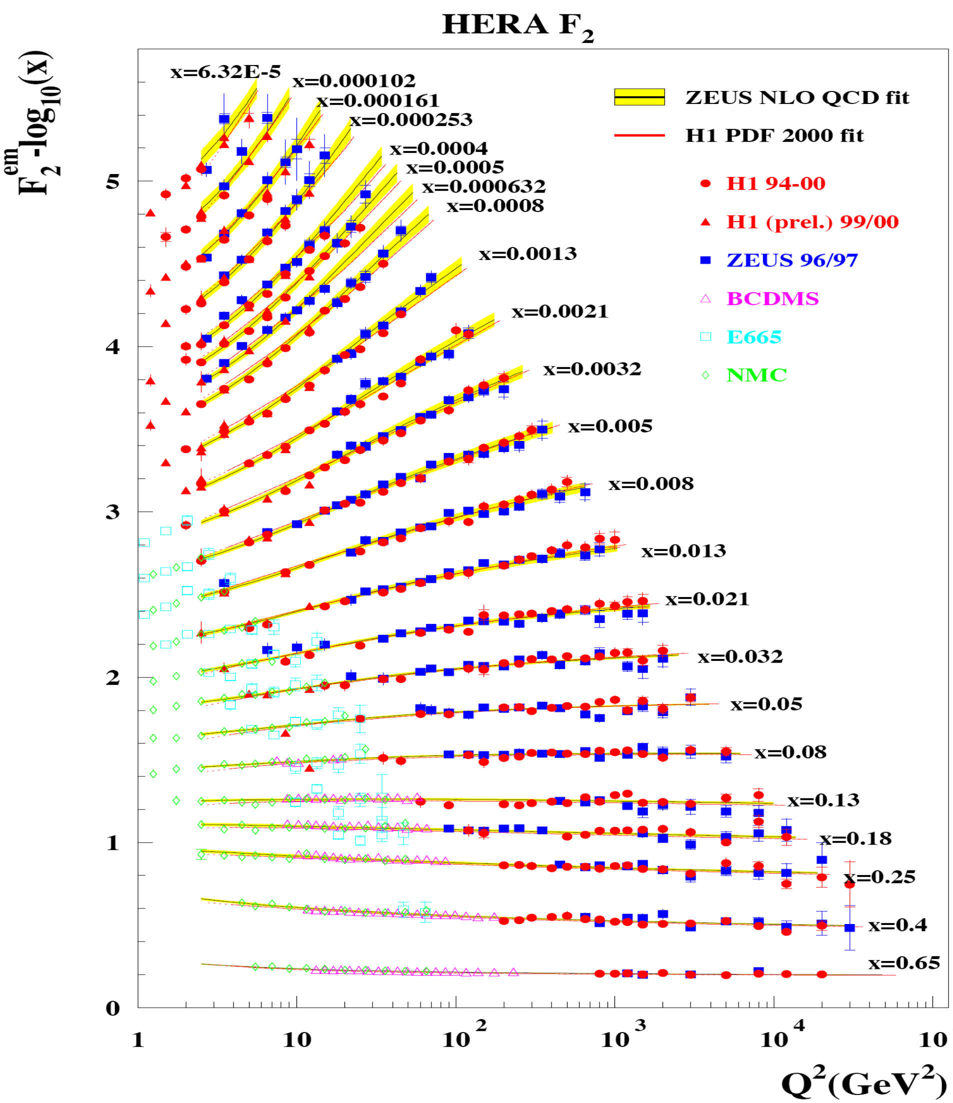
Jefferson Lab kinematics: $Q^2\approx M^2 \Rightarrow$ target mass effects, higher twist contributions and resonance excitations

- Non-zero $R=\frac{F_2}{2xF_1}\left(\frac{4M^2x^2}{Q^2}+1\right)-1$, $g_2^{HT}(x)=g_2(x)-g_2^{WW}(x)$
- Further Q^2-dependence (power series in $\frac{1}{Q^n}$)

Results

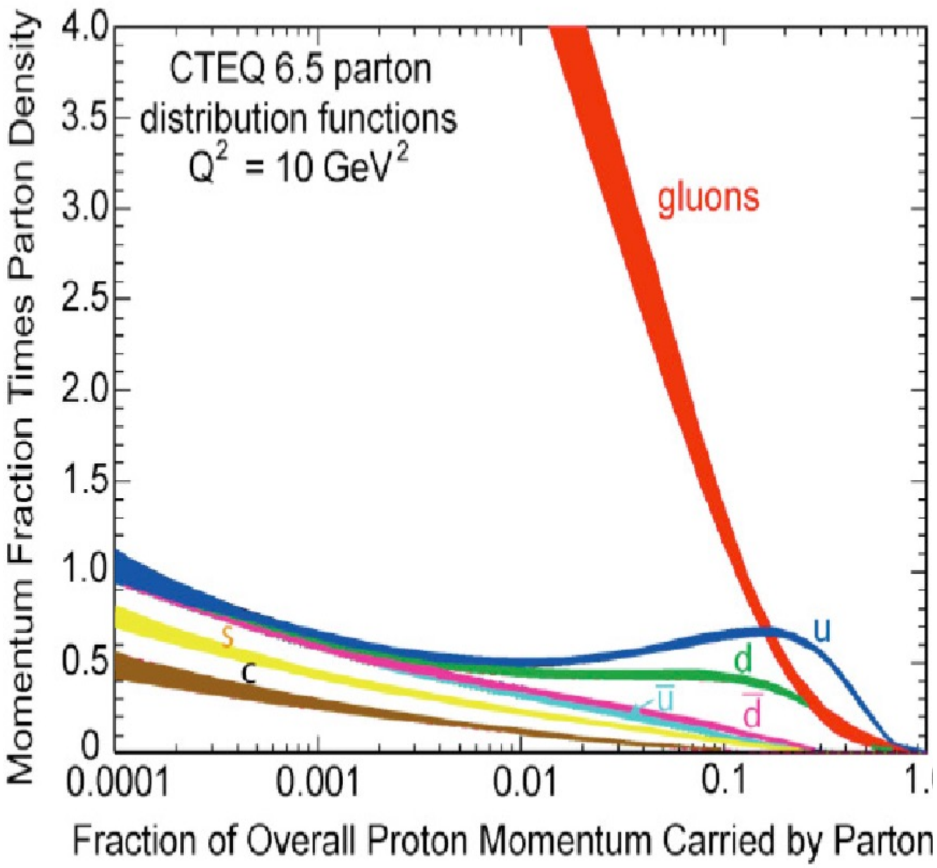


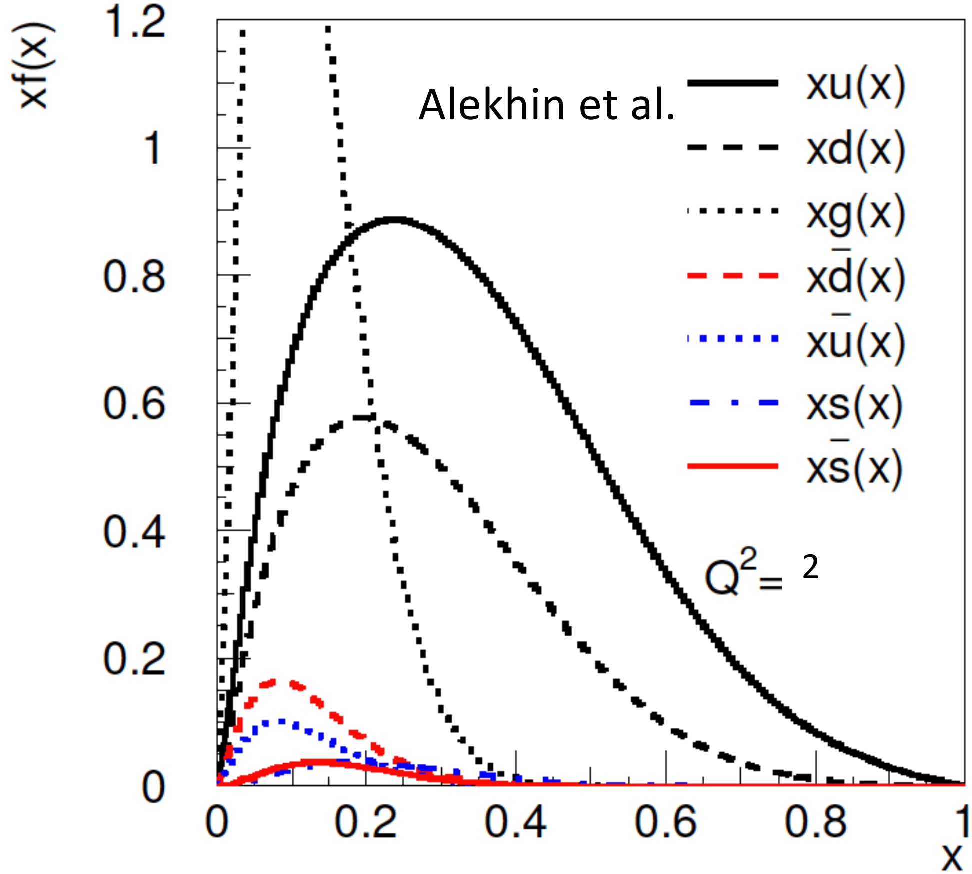
$F_2^p \cdot c$
GD11-P
⟨x⟩ c
0.008 1.6^{40}
0.011 1.6^{39}
0.015 1.6^{38}
0.019 1.6^{37}
0.025 1.6^{36}
0.033 1.6^{35}
0.040 1.6^{34}
0.049 1.6^{33}
0.060 1.6^{32}
0.073 1.6^{31}
0.089 1.6^{30}
0.108 1.6^{29}
0.134 1.6^{28}
0.166 1.6^{27}
0.211 1.6^{26}
0.273 1.6^{25}
0.366 1.6^{24}
0.509 1.6^{23}
0.679 1.6^{22}
HERA
JLAB
BCDMS
E665
NMC
SLAC
HERMES
Q^2 [GeV^2]

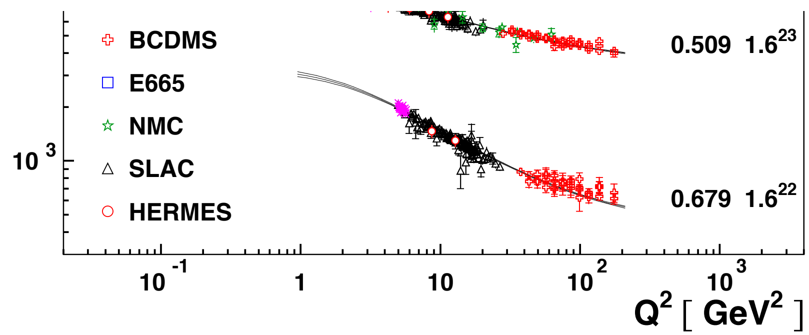


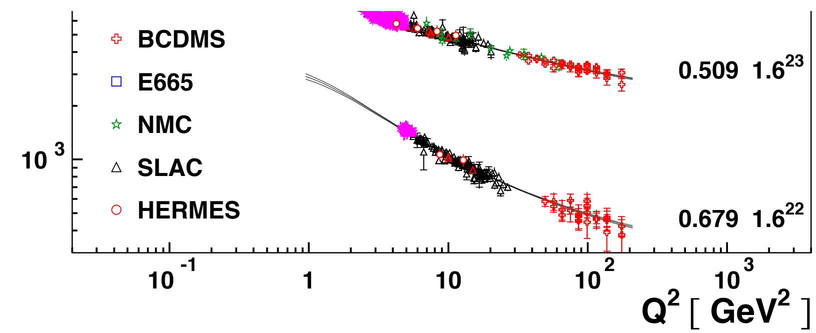
HERA F_2
F_2^{em}-$\log_{10}(x)$
x=6.32E-5
x=0.000102
x=0.000161
x=0.000253
x=0.0004
x=0.0005
x=0.000632
x=0.0008
x=0.0013
x=0.0021
x=0.0032
x=0.005
x=0.008
x=0.013
x=0.021
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.4
x=0.65
ZEUS NLO QCD fit
H1 PDF 2000 fit
H1 94-00
H1 (prel.) 99/00
ZEUS 96/97
BCDMS
E665
NMC
$Q^2(GeV^2)$

Unpolarized Structure Functions

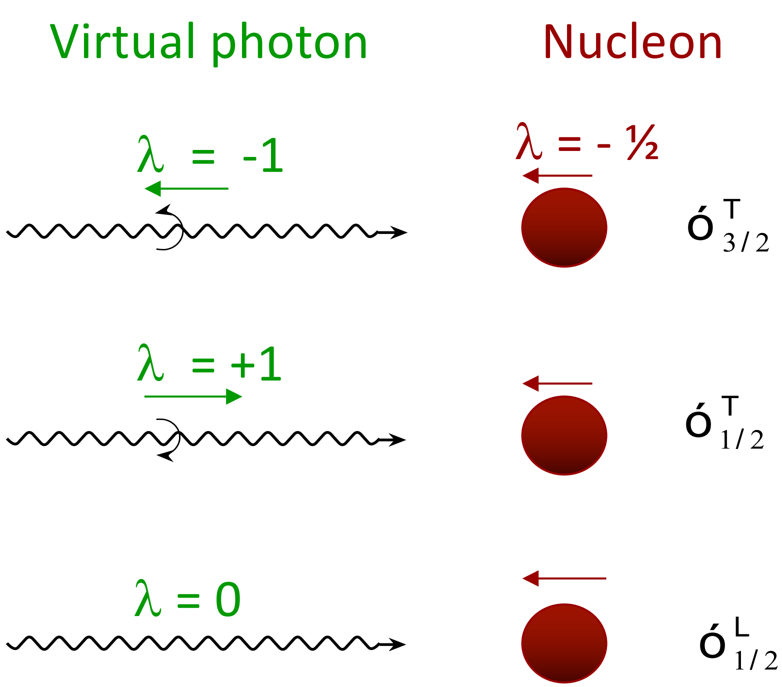


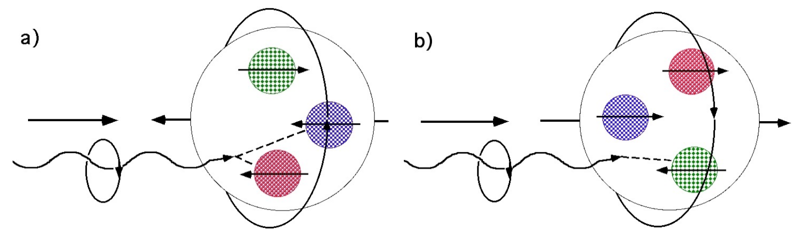






Virtual Photon Asymmetries

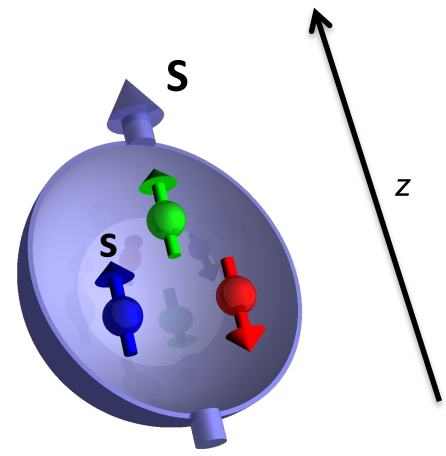




$$A_1 = \frac{\sigma_{1/2} - \sigma_{3/2}}{\sigma_T} \qquad A_2 = \frac{\sigma_{LT'}}{\sigma_T}$$

related to quark polarizations
$\Delta q/q$

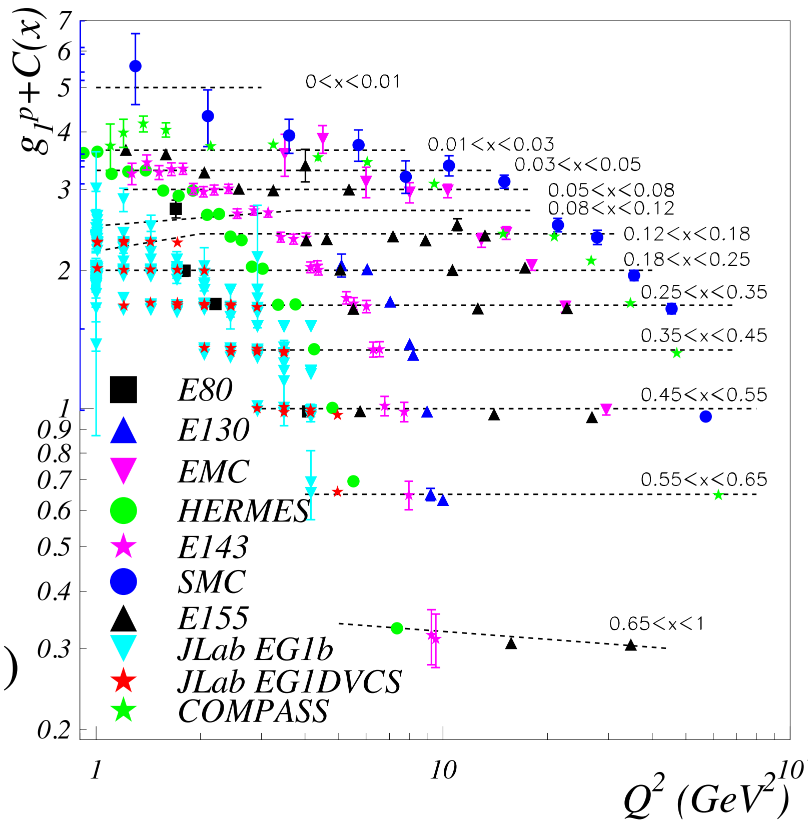
Polarized
Parton Distribution Functions



- Introduce two more quantities of interest:
  - Proton spin **S**
  - Parton spin **s**
  - Now we have 3 vectors: $\hat{z}, \vec{S}, \vec{s}$
  - **But**: Every observable must be a scalar
  - **And**: Spins are axial vectors!
  - **Finally**: Must treat longitudinal and transverse directions differently (boost)
  - 2 Pseudoscalars: $H = \vec{S} \cdot \hat{z}, h = \vec{s} \cdot \hat{z}$
  - 2 transverse (2D) axial vectors: $\vec{S}_\perp, \vec{s}_\perp$
  - 2nd Structure function

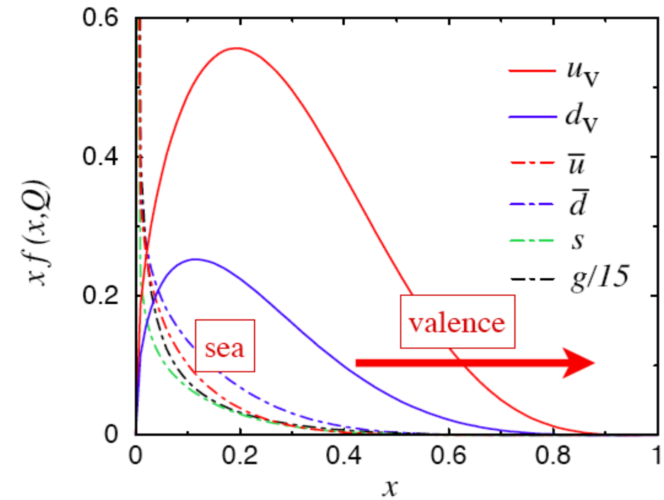
$$g_1^i(x) = \langle hH \rangle q_i(x) \text{ or } \langle hH \rangle G(x) = \Delta q_i(x) \text{ or } \Delta G(x)$$

$$\Delta q_i = \quad q \Uparrow\uparrow(x) - q \Uparrow\downarrow(x)$$

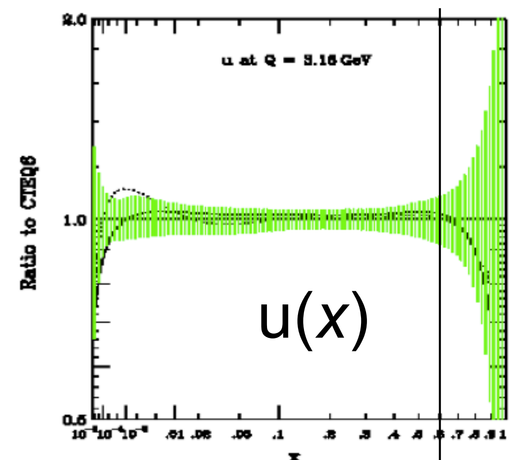


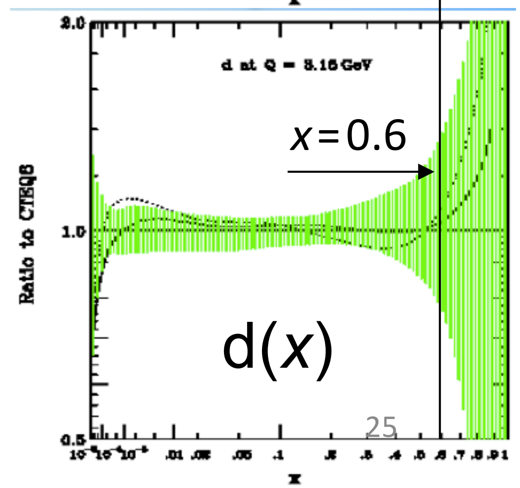
Can also form one more scalar: $T = \vec{S}_\perp \cdot \vec{s}_\perp$ (not measurable in DIS) → Transversity $h_1(x)$

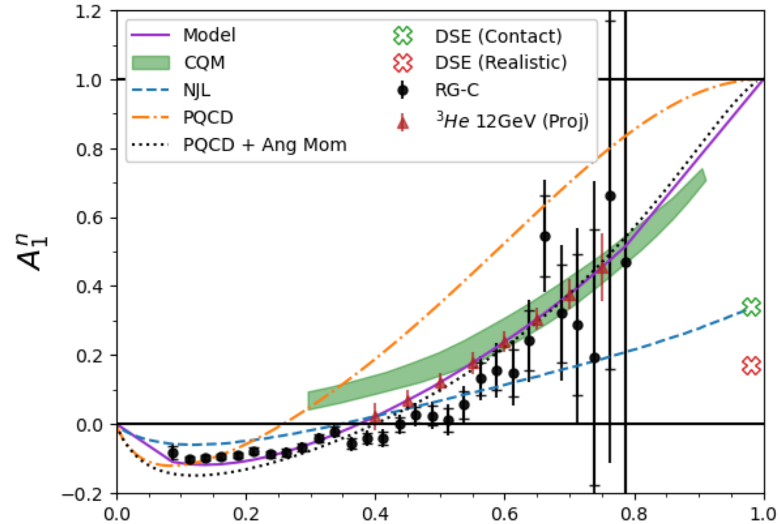
Valence PDFs

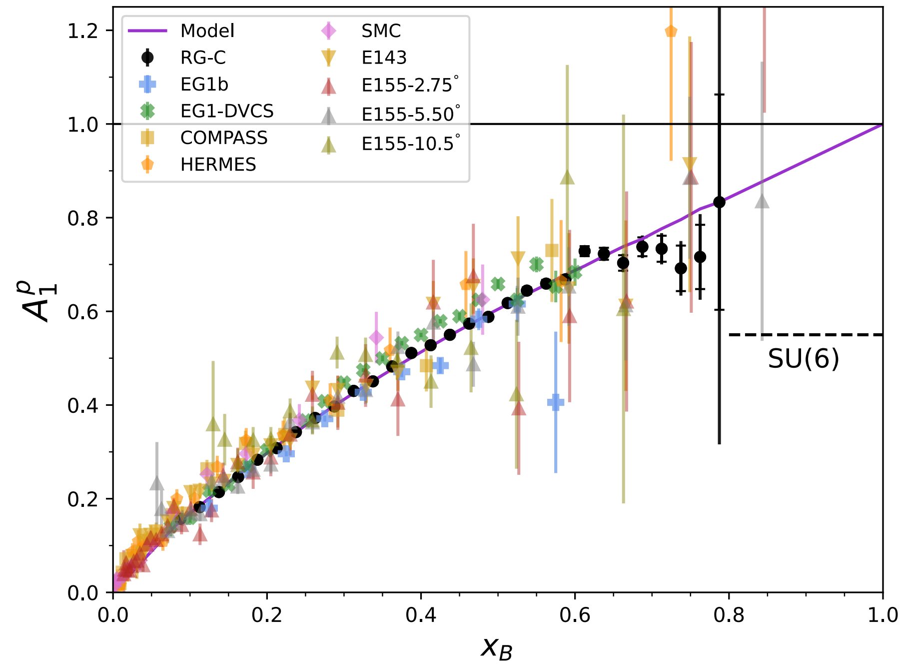


- Behavior of PDFs still unknown for $x \to 1$
  - SU(6): d/u = 1/2, Δu/u = 2/3, Δd/d = -1/3 for all x
  - Relativistic Quark model: Δu, Δd reduced
  - Hyperfine effect (1-gluon-exchange): Spectator spin 1 suppressed, d/u → 0, Δu/u → 1, Δd/d → -1/3
  - Helicity conservation: d/u → 1/5, Δu/u → 1, Δd/d → 1
  - Orbital angular momentum: can explain slower convergence to Δd/d → 1
- Plenty of data on proton → mostly constraints on u and Δu
- Knowledge on d limited by lack of free neutron target (nuclear binding effects in d, ^{3}He)
- Large x requires very high luminosity and resolution; binding effects become dominant uncertainty for the neutron



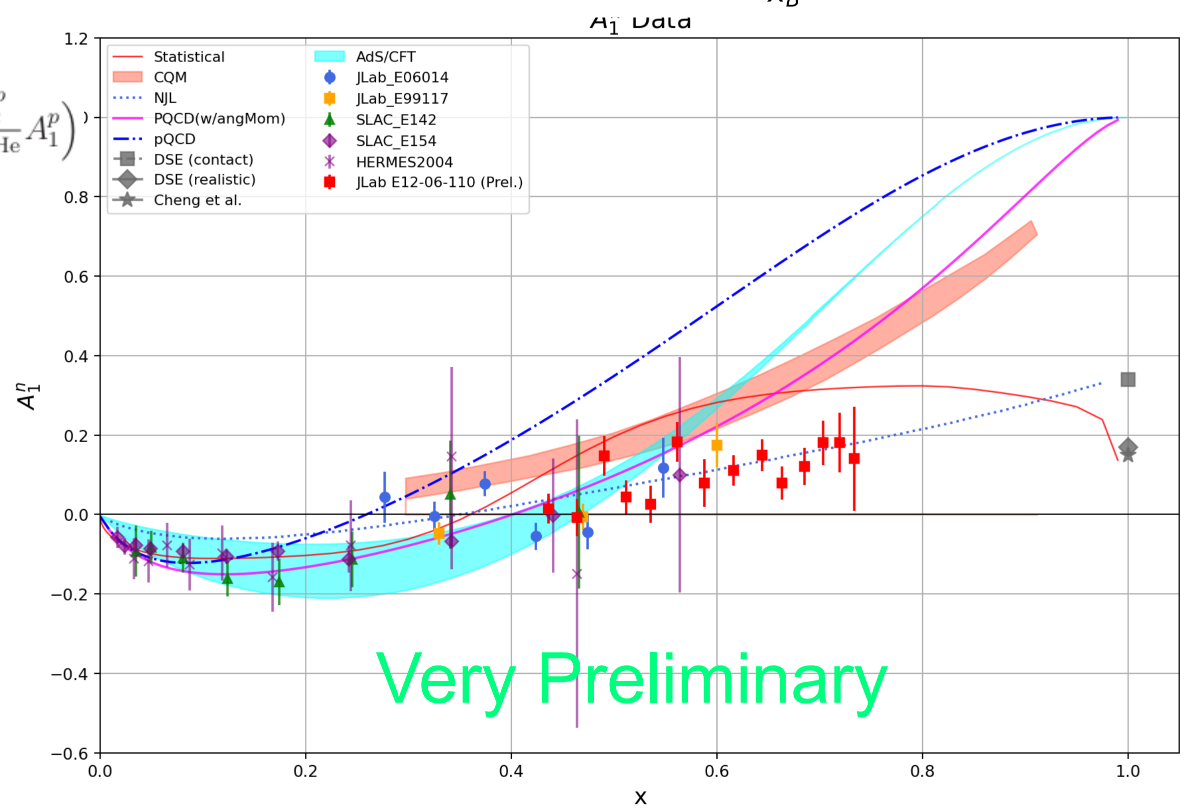


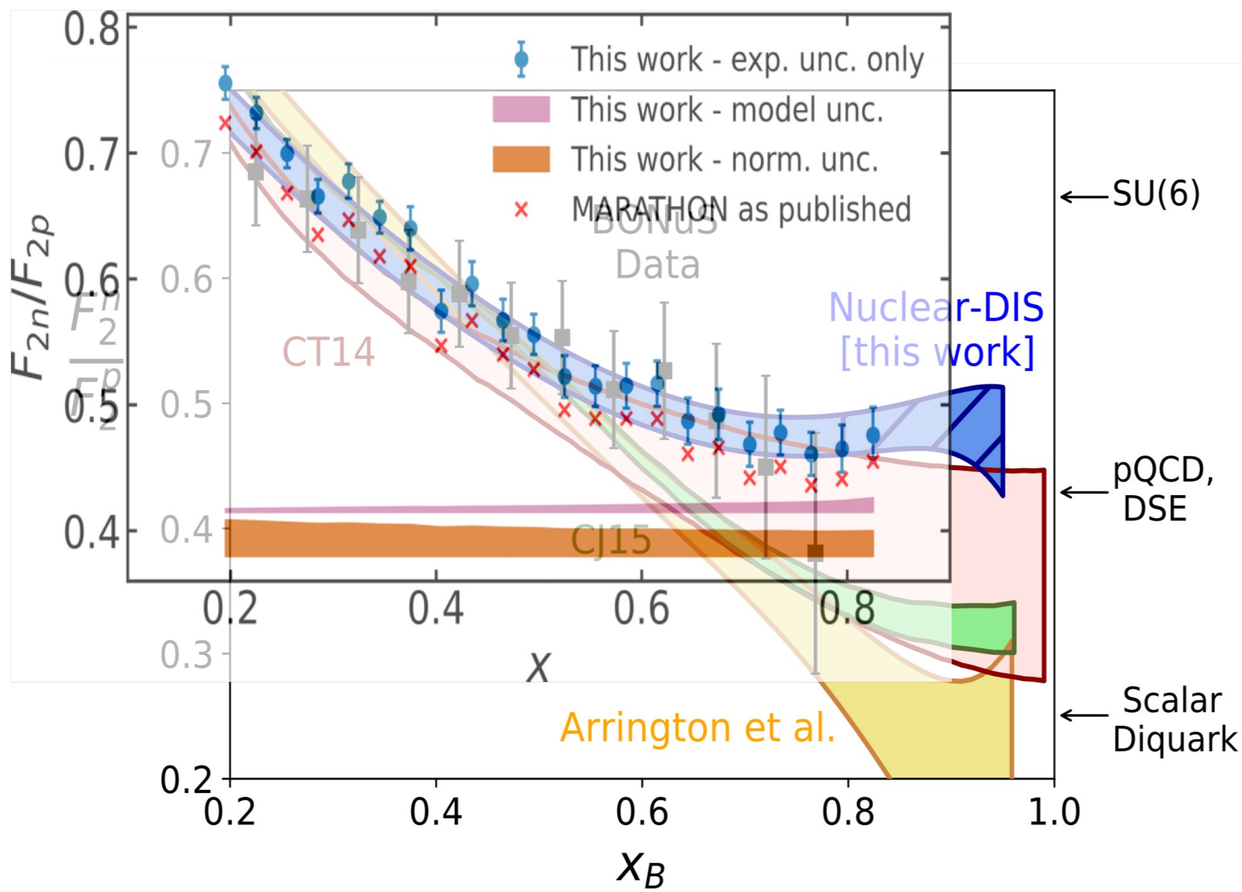


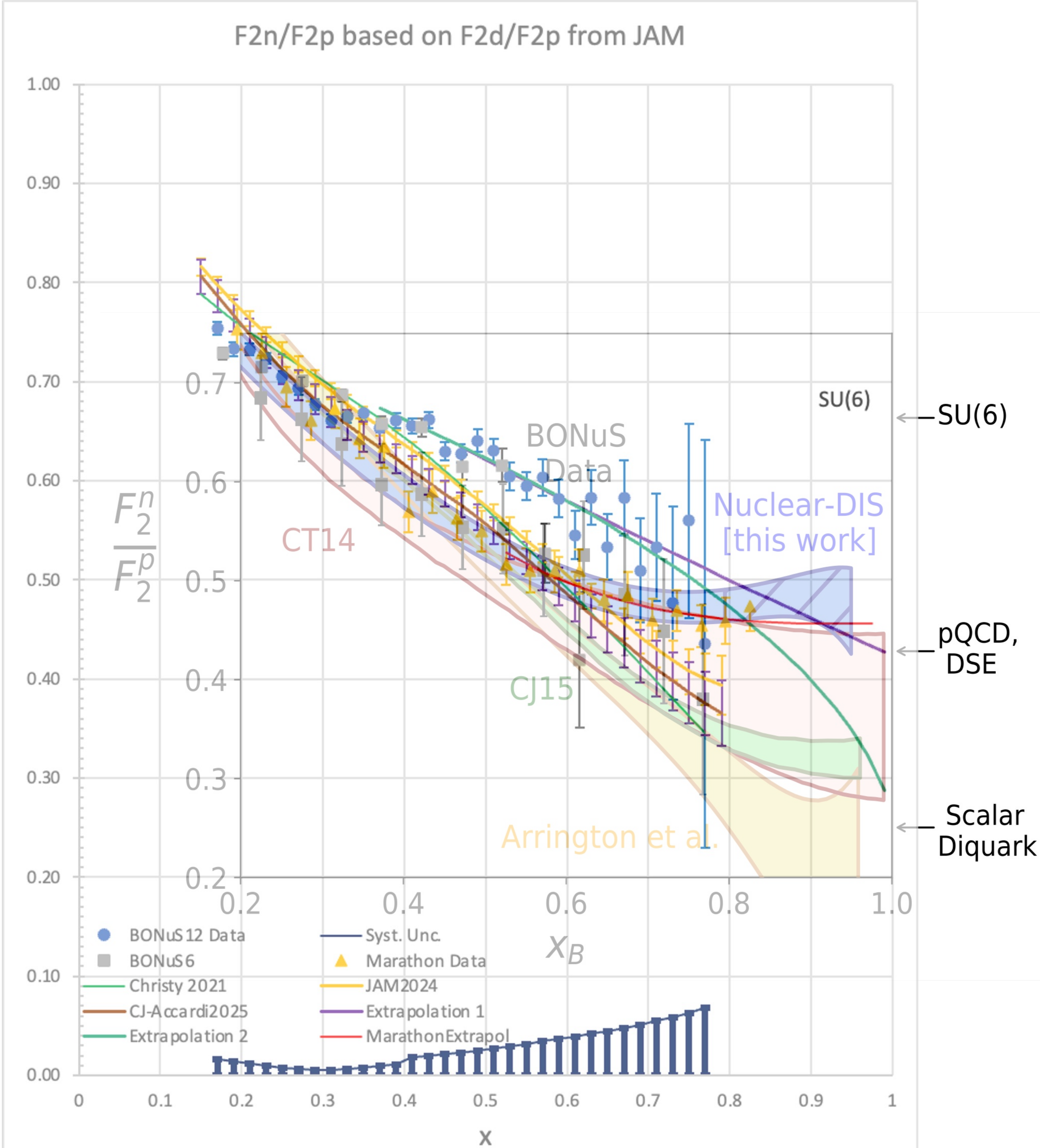


A1n extracted using effective polarizations plus our own 3He fit)

$$A_1^n = \frac{1}{P_n}\frac{F_2^{^3\mathrm{He}}}{F_2^n}\left(A_1^{^3\mathrm{He}} - 2P_p\frac{F_2^p}{F_2^{^3\mathrm{He}}}A_1^p\right)$$




This work - exp. unc. only
This work - model unc.
This work - norm. unc.
MARATHON as published
BONuS
Data
Nuclear-DIS
[this work]
CT14
CJ15
Arrington et al.
SU(6)
pQCD,
DSE
Scalar
Diquark
F_{2n}/F_{2p}
x
x_B


F2n/F2p based on F2d/F2p from JAM
BONuS Data
Nuclear-DIS [this work]
CT14
CJ15
Arrington et al.
SU(6)
pQCD, DSE
Scalar Diquark
BONuS12 Data
BONuS6
Christy 2021
CJ-Accardi2025
Extrapolation 2
Syst. Unc.
Marathon Data
JAM2024
Extrapolation 1
MarathonExtrapol
x
x_B
$\frac{F_2^n}{F_2^p}$