Motion of Projectiles and Charged Particles

Newton’s Laws

Newton’s first law
In the absence of forces, a particle moves with constant velocity.

Newton’s second law
For any particle of mass \( m \), the net force \( F \) on the particle is always equal to the mass \( m \) times the particle’s acceleration:

\[
F = m \ddot{a}
\]

Newton’s third law
If object 1 exerts a force \( F_{21} \) on object 2, then object 2 always exerts a reaction force \( F_{12} \) on object 1 given by

\[
F_{12} = -F_{21}
\]

Newton’s Second Law

Newton’s second law as a differential equation

\[
m \frac{d^2 \vec{r}}{dt^2} = \vec{F}
\]

or in the Cartesian coordinates

\[
m \frac{d^2 x}{dt^2} = F_x
\]

\[
m \frac{d^2 y}{dt^2} = F_y
\]

\[
m \frac{d^2 z}{dt^2} = F_z
\]

Air resistance

The physical origin of the terms:

The linear term corresponds to the viscosity drag of the medium

The quadratic term describes the acceleration of the mass of air pushed by the projectile

for medium to fast speeds \( f_{sa} \ll f_{quad} \)

Drag force at fast speed (Rayleigh’s equation)

\[
\vec{F}_D = -\frac{1}{2} C \rho A v^2 \vec{v}
\]

\( C \) – drag coefficient (dimensionless constant)

\( \rho \) – air density

\( A \) – projectile cross section (area)

\( v \) – speed

Motion in \((x,y)\) plane

Equations of motion with gravitational and drag forces

\[
m \frac{d^2 x}{dt^2} = F_{dx}
\]

\[
m \frac{d^2 y}{dt^2} = -mg + F_{dy}
\]

Imposing initial conditions the system of ordinary differential equations can be solved numerically using methods for solving ODE initial value problem

Runge-Kutta method normally works well for the system
Drag coefficient \( C \)

Drag coefficient \( C \) depends on an object shape and can be determined by wind tunnel measurements. For many objects it can be approximated by a value within 0.05 – 1.0.

\[
\vec{F}_D = -\frac{1}{2} C \rho A \vec{v} = -D \vec{v}
\]

Terminal speed (approx.)

\[
mg = \frac{1}{2} C \rho A \vec{v}^2
\]

<table>
<thead>
<tr>
<th>object</th>
<th>speed (m/s)</th>
<th>speed (mph)</th>
<th>distance (m) 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>shot</td>
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<td>316</td>
<td>2500</td>
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<tr>
<td>sky diver</td>
<td>60</td>
<td>130</td>
<td>430</td>
</tr>
<tr>
<td>baseball</td>
<td>42</td>
<td>92</td>
<td>210</td>
</tr>
<tr>
<td>basketball</td>
<td>20</td>
<td>44</td>
<td>47</td>
</tr>
<tr>
<td>raindrop</td>
<td>7</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>parachutist</td>
<td>5</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>

References


C. Lindsey et al The Physics and Technology of Tennis, USRSA (2004)


Aerodynamic drag crisis

\[
\vec{F}_D = -\frac{1}{2} C(\vec{v}) \rho A \vec{v}
\]

Generally \( C \) depends on speed \( v \) (aerodynamic drag crisis)

\[
R = \frac{vD}{\nu} = \frac{vD}{v_dR}
\]

\( v \) – velocity, \( d \) – diameter,
\( \nu \) – kinematic viscosity of air

System of equations for calculations

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= -D v_x / m \\
\frac{d^2 y}{dt^2} &= -g - D v_y / m \\
\end{align*}
\]

where \( D = \frac{1}{2} C \rho A \)

since air density varies with altitude, one may approximate it as

\[
\rho = \rho_0 \exp(-y / y_0)
\]

where \( \rho_0 \) is the density as sea level (\( y = 0 \)).
the coefficient \( y_0 \) is about 10,000 m

Drag force in \((x,y)\) plane

\[
\begin{align*}
\vec{F}_{Dx} &= -D v_x \cos(\theta) = -D v_x \sqrt{v_x^2 + v_y^2} \\
\vec{F}_{Dy} &= -D v_y \sin(\theta) = -D v_y \sqrt{v_x^2 + v_y^2} \\
\end{align*}
\]
A good test case

Attention!
A good way to test your computer code.
The system has an analytic solution for $D=0$

$$x = x_0 + v_0 \cos(\theta_0) t$$
$$v_x = v_0 \cos(\theta_0)$$

$$y = y_0 + v_0 \sin(\theta_0) t - \frac{gt^2}{2}$$
$$v_y = v_0 \sin(\theta_0) - gt$$

and the equation of the path is

$$y = y_0 + x \frac{\sin \theta_0}{\cos \theta_0} + \frac{g}{2} \frac{x^2}{v_0^2 \cos^2 \theta_0}$$

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Effect of spin (Magnus force)

$$\vec{F}_M = S(v) \vec{\omega} \times \vec{v}$$

common approximation

$$\frac{F_M}{m} = S_0 \vec{\omega} \vec{v}$$

coefficients for $S_0$ can be found elsewhere

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Effect of wind (parallel to the ground)

let $w$ is the speed of wind, then

+ $w$ corresponds to tailwind
- $w$ corresponds to headwind

$$v_x = u_x + w$$
$$v_y = u_y$$

$$v = \sqrt{(u_x + w)^2 + u_y^2}$$

$$\frac{d^2 x}{dt^2} = -\frac{D(u_x + w)v}{m}$$

$$\frac{d^2 y}{dt^2} = -g - \frac{D}{m} u_y v$$
Lorentz force on a charged particle

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

for motion in crossed fields

\[ \mathbf{E} \perp \mathbf{B} \quad \mathbf{E} = (E_x, E_y, 0) \quad \mathbf{B} = (0, 0, B_z) \]

\[
\begin{align*}
    m \frac{d^2x}{dt^2} &= E_x + qB_y v_y \\
    m \frac{d^2y}{dt^2} &= E_y - qB_x v_x + \frac{q\mathbf{B}}{m} \Rightarrow \frac{d^2x}{dt^2} &= \frac{E_x}{m} + \omega v_y \\
    m \frac{d^2z}{dt^2} &= 0 \Rightarrow \frac{d^2z}{dt^2} = 0
\end{align*}
\]

Possible projects

Projectile motion in sport
Motion in electric and magnetic fields
Laser cooling and trapping