Chapter 3

Motion in Two or Three Dimensions

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Outline

1. Position, velocity, acceleration
2. Motion in a plane (Set of equations)
3. Projectile Motion (Range, Height, Velocity, Trajectory)
4. Circular Motion (Polar coordinates, Time derivatives)
5. Relative Motion

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Position, velocity, acceleration

1. Position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

2. Average velocity $\vec{v}_{av} = \dfrac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$

3. Instantaneous velocity $\vec{v} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$

$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ $\quad v = \sqrt{v_x^2 + v_y^2 + v_z^2}$

4. Average acceleration $\vec{a}_{av} = \dfrac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$

5. Instantaneous acceleration $\vec{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{v}}{\Delta t} = \dfrac{d\vec{v}}{dt}$

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Part 1

Motion in a plane

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Motion in Plane When Acceleration is Constant

- Consider a case when acceleration in a plane is constant $\vec{a} = \vec{A} = \text{const}$
- In 2-D Cartesian coordinates, this can be written as

$$a_x\,\hat{\imath} + a_y\,\hat{\jmath} = A_x\,\hat{\imath} + A_y\,\hat{\jmath}$$

- Or, in terms of velocity:

$$\frac{dv_x}{dt}\,\hat{\imath} + \frac{dv_y}{dt}\,\hat{\jmath} = A_x\,\hat{\imath} + A_y\,\hat{\jmath}$$

- Since $\hat{\imath}$ and $\hat{\jmath}$ are linearly independent, it is clear that

$$\frac{dv_x}{dt} = A_x \quad , \quad \frac{dv_y}{dt} = A_y$$

Motion that occurs in x -direction is completely independent
of motion that occurs in the y-direction.

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Set of equations

- Each of these equations can be integrated to obtain components of velocity and position

$$\begin{cases} a_x &= A_x \\ v_x &= A_x t + v_{x0} \\ x &= \frac{1}{2}A_x t^2 + v_{x0}t + x_0 \end{cases}$$

$$\begin{cases} a_y &= A_y \\ v_y &= A_y t + v_{y0} \\ y &= \frac{1}{2}A_y t^2 + v_{y0}t + y_0 \end{cases}$$

Anything that can possibly be known about
motion in the xy-plane at constant acceleration is contained within these equations

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Being practical: 2D motion with const. a

1. initial value problem:
   knowing initial conditions (x_0,y_0 & v_{x0},v_{y0}) and acceleration (a_x,a_y) one may find position (x,y) and velocity (v_x,v_y) at any moment in time.
2. final value problem:
   when some (or all) or final values are known, one may find required initial conditions to satisfy the final values
3. a mixture of initial and final value problems (just algebra :-)

$$\begin{cases} v_x = v_{x0} + a_x t \\ x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \\ v_y = v_{y0} + a_y t \\ y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2 \end{cases}$$

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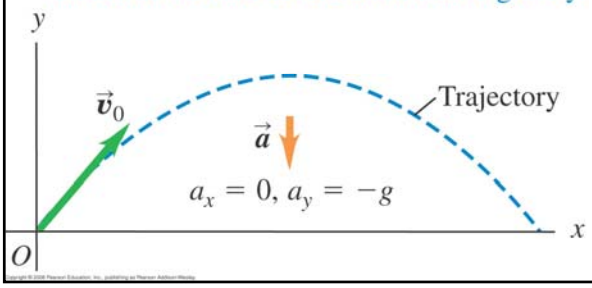
Part 2

Projectile Motion

- A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
- Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.



$$\begin{cases} v_x = v_{x0} \\ x = x_0 + v_{x0} t \\ v_y = v_{y0} - g_y t \\ y = y_0 + v_{y0} t - \frac{1}{2} g t^2 \end{cases}$$

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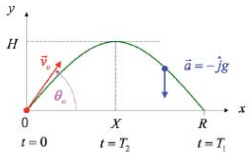
Projectile Motion – 2 D Example

- Projectile is launched at angle θ_0



Note that

$v_{x0} = v_0 \cos\theta_0$

$v_{y0} = v_0 \sin\theta_0$

- Two sets of equations for motion in x-direction and y-direction can now be written as

$$\begin{cases} a_x &= A_x = 0 \\ v_x &= v_{x0} = v_0 \cos\theta_0 \\ x &= v_{x0}t = v_0 t \cos\theta_0 \end{cases} \qquad \begin{cases} a_y &= A_y = -g \\ v_y &= -gt + v_{y0} \\ &= -gt + v_0 \sin\theta_0 \\ y &= -\frac{1}{2}gt^2 + v_{y0}t \sin\theta_0 \end{cases}$$

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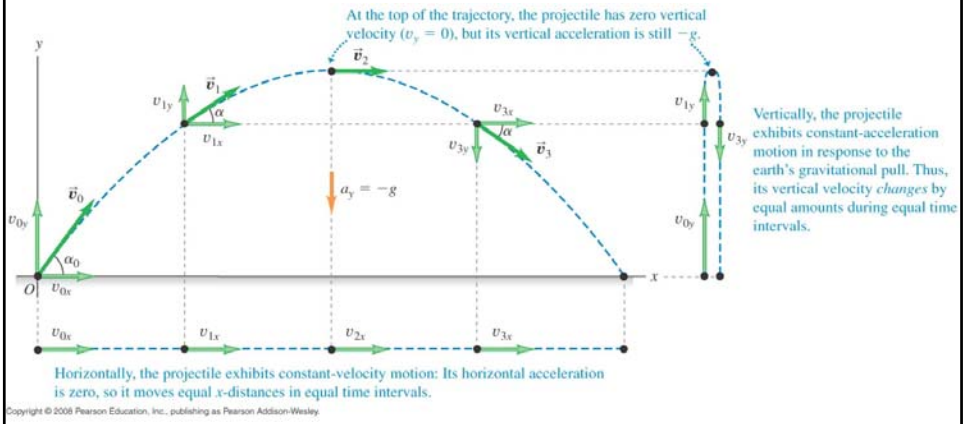
Projectile Motion – 2 D



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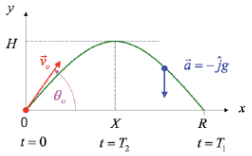
Range of Projectile



Note that

$x(T_1) = R$

$y(T_1) = 0$

Using equations for y and x:

$0 = -\frac{1}{2}gT_1^2 + v_0 T_1 \sin\theta_0 \quad , \quad R = v_0 T_1 \cos\theta_0$

From 1st equation:

$T_1 = 2\frac{v_0}{g}\sin\theta_0$

Substituting T_1 into 2nd equation:

$R = 2\frac{v_0^2}{g}\sin\theta_0\cos\theta_0 = \frac{v_0^2}{g}\sin(2\theta_0)$

Maximal range is reached when

$\sin(2\theta_0) = 1$

$\Rightarrow R_{\max} = v_0^2/g$ for $\theta_0 = 45^\circ$

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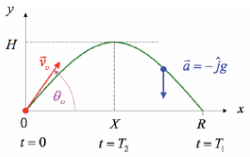
Maximum height reached by projectile



At maximum height:

$y(T_2) = H$

$v_y(T_2) = 0$

Using equations for v_y and y:

$0 = -gT_2 + v_0 \sin\theta_0 \quad , \quad H = -\frac{1}{2}gT_2^2 + v_0 T_2 \sin\theta_0$

From 1st equation:

$T_2 = \frac{v_0}{g}\sin\theta_0 = \frac{1}{2}T_1$

Substituting T_2 into 2nd equation:

$H = \frac{v_0^2}{2g}\sin^2\theta_0 = \frac{v_{y0}^2}{2g}$

Maximum height location $x(T_2) \equiv X$ from x-equation

$X = v_0 T_2 \cos\theta_0 = \frac{v_0^2}{g}\cos\theta_0\sin\theta_0 = \frac{R}{2}$

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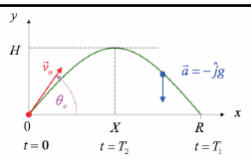
Velocity at the ground



Impact occurs at time $t = T_2 \Rightarrow$
velocity components at that time are
$v_x = v_0 \cos\theta_0 \quad , \quad v_y = -gT_1 + v_0 \sin\theta_0 = -v_0 \sin\theta_0$

Speed of projectile at impact:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(v_0 \cos\theta_0)^2 + (-v_0 \sin\theta_0)^2} = v_0$$

Surprised?
This result will be obvious later when we learn about conservation of energy

Trajectory of the projectile

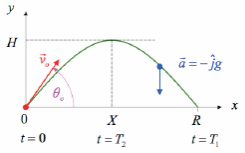
To find y as function of x extract t from equation for $x(t)$:

$$x = v_0 t \cos\theta_0 \Rightarrow t = \frac{x}{v_0 \cos\theta_0}$$

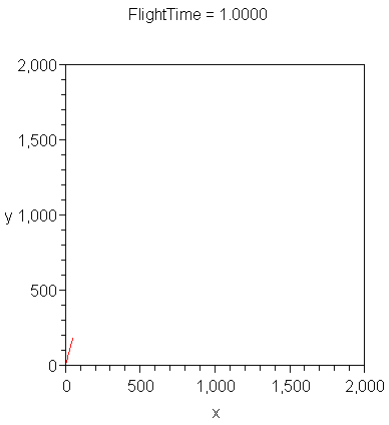
and substitute t into equation for y:

$$\begin{aligned} y &= -\frac{1}{2}gt^2 + v_{y0}\, t \sin\theta_0 \\ &= -\frac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2 + v_0 \sin\theta_0 \left(\frac{x}{v_0 \cos\theta_0}\right) \\ &= \underbrace{\left(-\frac{g}{2v_0^2 \cos^2\theta_0}\right)}_{-a} x^2 + \underbrace{(\tan\theta_0)}_{b}\, x \end{aligned}$$

Equation has the form $y = -ax^2 + b$
This is the equation of a parabola.



example 1: initial value problem



http://www.physics.odu.edu/~godunov/teaching/phys420/index.html

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Using parabola equation

- The x-coordinate of the point where y has the highest value (its maximum value) can be found using

$$\left(\frac{dy}{dx}\right)\Big|_{x=X} = -g\frac{X}{v_0^2\cos^2\theta_0} + \tan\theta_0 = 0\ ,$$

$$\Rightarrow X = \frac{v_0^2}{g}\sin\theta_0\cos\theta_0 = \frac{R}{2}$$

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Problem

$$\begin{cases} v_x &= v_{x0} \\ x &= x_0 + v_{x0}t \\ v_y &= v_{y0} - g_y t \\ y &= y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{cases}$$

Figure shows a pirate ship 560 m from a fort defending the harbor entrance of an island. A defense cannon, located at sea level, fires balls at initial speed v_0=82 m/s.

(a) at what angle from the horizontal must a ball be fired to hit the ship?

$x_f = 560.0\ m$

$v_0 = 82.0\ m/s$

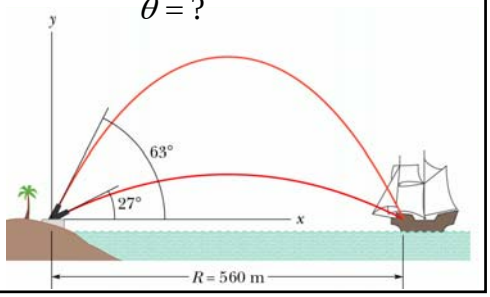
$g = 9.81\ m/s^2$

$\theta = ?$

$$R = \frac{v_0^2}{g}\sin(2\theta)$$

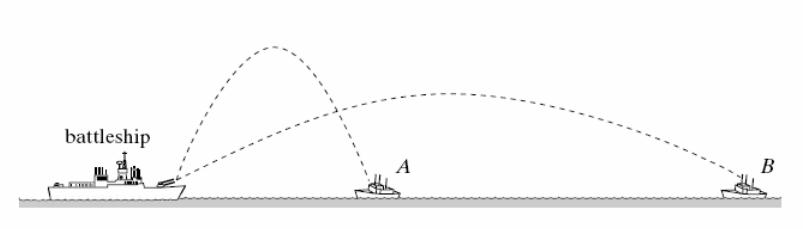
two solutions:

27^0 and 63^0.

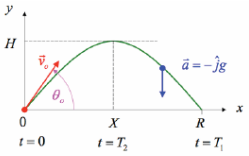


Question

A battleship simultaneously fires two shells toward two enemy ships, one close by (A), one far away (B). The shells leave the battleship at different angles and travel along the parabolic trajectories indicated below. Which of the two enemy ships gets hit first? Do you need more information to answer the question?



Maximum height reached by projectile



At maximum height:

$$y(T_2) = H$$
$$v_y(T_2) = 0$$

Using equations for v_y and y:

$$0 = -gT_2 + v_0 \sin\theta_0 \quad , \quad H = -\tfrac{1}{2}gT_2^2 + v_0 T_2 \sin\theta_0$$

From 1st equation:

$$T_2 = \frac{v_0}{g}\sin\theta_0 = \frac{1}{2}T_1$$

Substituting T_2 into 2nd equation:

$$H = \frac{v_0^2}{2g}\sin^2\theta_0 = \frac{v_{y0}^2}{2g}$$

Maximum height location $x(T_2) \equiv X$ from x-equation

$$X = v_0 T_2 \cos\theta_0 = \frac{v_0^2}{g}\cos\theta_0 \sin\theta_0 = \frac{R}{2}$$

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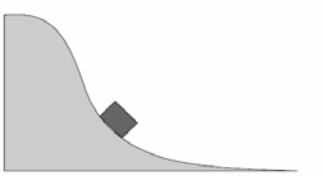
Question

A cart on a roller-coaster rolls down the track shown below. As the cart rolls beyond the point shown, what happens to its speed and acceleration in the direction of motion?

1. Both decrease.
2. The speed decreases, but the acceleration increases.
3. Both remain constant.
4. The speed increases, but acceleration decreases.
5. Both increase.
6. Other



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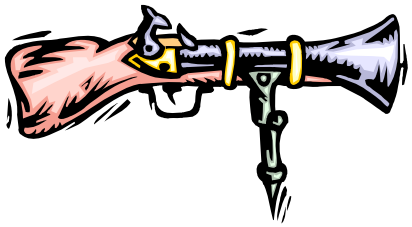
Question

At the same instant that you fire a bullet horizontally from a gun, you drop a bullet from the height of the barrel. If there is no air resistance, which bullet hits the ground first?

$$\begin{cases} v_x &= v_{x0} \\ x &= x_0 + v_{x0}t \\ v_y &= v_{y0} - g_y t \\ y &= y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{cases}$$



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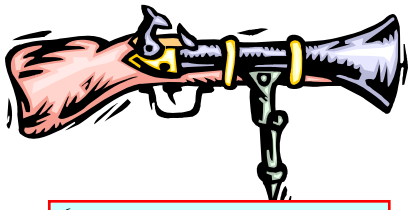
Question

When a rifle is fired at a distant target, the barrel is not lined up exactly on the target. Why not?



$$\begin{cases} v_x &= v_{x0} \\ x &= x_0 + v_{x0}t \\ v_y &= v_{y0} - g_y t \\ y &= y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{cases}$$

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Problem

A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side the car is on is 21.3 m above the river, while the opposite side is a mere 1.8 m above the river. The river itself is a raging torrent 61.0 wide.

How fast should the car be traveling just as it leaves the cliff in order just to clear the river and land safely on the opposite side?

What is the speed of the car just before it lands safely on the other side?

Problem (cont)

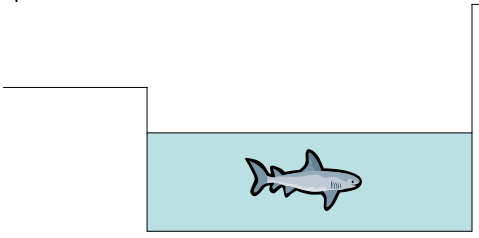
x_i = 0.0 m y_i = 21.3 m

x_f = 61 m y_f = 1.8 m

v_0 = ?

v_f = ?

$$\begin{cases} v_x = v_{x0} \\ x = x_0 + v_{x0}t \\ v_y = -g_y t \\ y = y_0 - \frac{1}{2}gt^2 \end{cases}$$



1. Use vertical motion to find time in the air (last equation)
2. Then find v_0 from the second equation

Problem 3.55

According to the Guinness Book of World Records, the longest home run ever measured was hit by Roy "Dizzy" Carlyle in a minor league game. The ball traveled 188 m (618 ft) before landing on the ground outside the ballpark.

Assuming the ball's initial velocity was 45^0 above the horizontal and ignoring air resistance, what did the initial speed of the ball need to be to produce such a home run if the ball was hit at a point 0.9 m (3.0 ft) above ground level? Assume that the ground was perfectly flat.

How far would the ball be above a fence 3.0 m (10 ft) high if the fence was 116 m (380 ft) from home plate?

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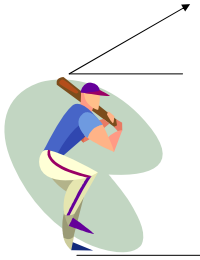
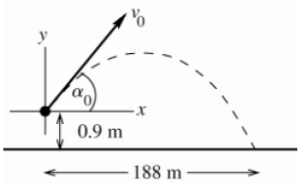
Problem (cont)

$x_0 = 0$ $\quad y_0 = 0.9$ m

$x_f = 188$ m, $\quad y_f = 0.0$ m

$v_0 = ?$





$$\begin{cases} v_x &= v_{x0} \\ x &= x_0 + v_{x0}t \\ v_y &= v_{y0} - g_y t \\ y &= y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{cases}$$

1. find time for horizontal motion from the second equation
2. using this time find v_0 from the last equation
3. a fence – then find time to fly to the fence in x direction and then y position

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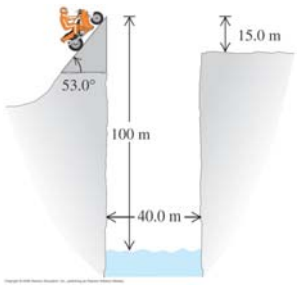
Problem 3.63

A physics professor did daredevil stunts in his spare time. His last stunt was an attempt to jump across a river on a motorcycle. The takeoff ramp was inclined at 53.0^0 , the river was 40.0 m wide, and the far bank was 15.0 m lower than the top of the ramp. The river itself was 100 m below the ramp. You can ignore air resistance.

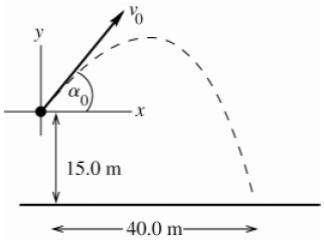


a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank?

b) If his speed was only half the value found in A, where did he land?

Problem 3.63

Data



$$x = x_0 + v_0 \cos\alpha_0 \cdot t$$

$$y = y_0 + v_0 \sin\alpha_0 \cdot t - \frac{1}{2} g t^2$$

a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank?

b) If his speed was only half the value found in A, where did he land?

Given: $(x, x_0), (y, y_0), \alpha_0$

Unknown: v_0, t

a) solve the system

b) find vertical time (now v_0 is given) and then the horizontal distance

How far and how high?

The fastest man (100 meters): 9.74 s
Asafa Powell (Jamaica) Rieti, Italy September 9, 2007

How high can he jump if … (world record 2.45 m)

How far can he jump if … (world record 8.95 m)

$$h = \frac{v^2}{2g}$$

$$R = \frac{v^2}{g}$$

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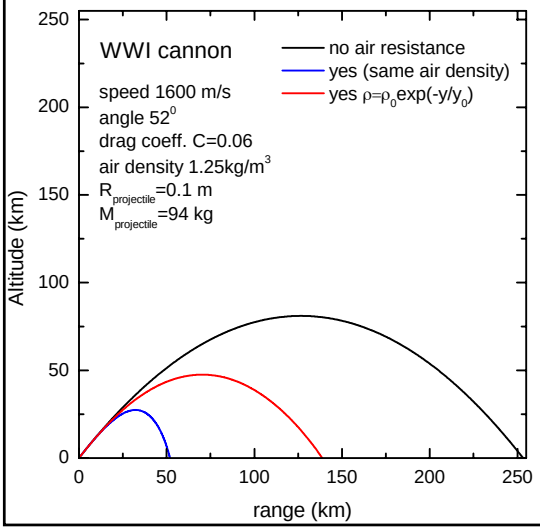
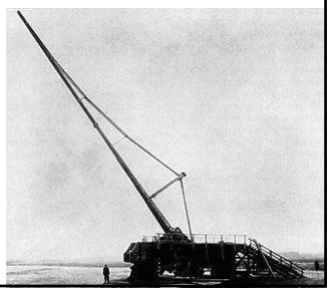
Going beyond of simple projectile motion

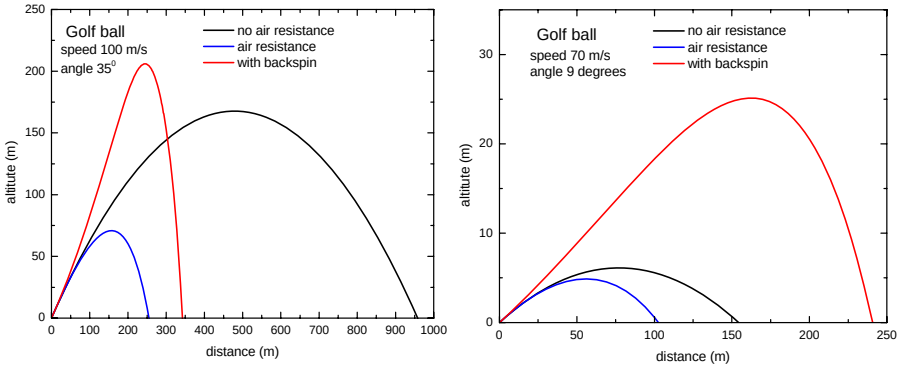
W W I – the largest cannon: effect of air resistance





Going beyond of simple projectile motion

effect of air resistance – the acceleration is no longer constant



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Going beyond of simple projectile motion

effect of air resistance – the acceleration is no longer constant

terminal velocity

$$mg = \frac{1}{2} C\rho A v^2$$

object	speed (m/s)	speed (mph)	distance (m) 95%
shot	145	316	2500
sky diver	60	130	430
baseball	42	92	210
basketball	20	44	47
raindrop	7	15	6
parachutist	5	11	3

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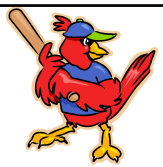
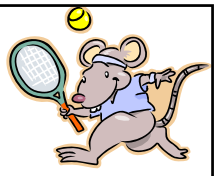
References





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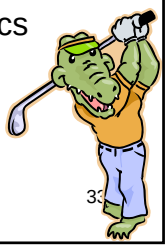

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Part 3

Uniform Circular Motion

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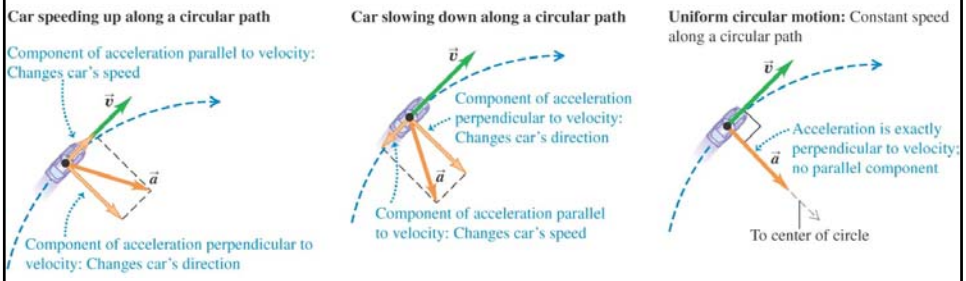
Motion in a circle



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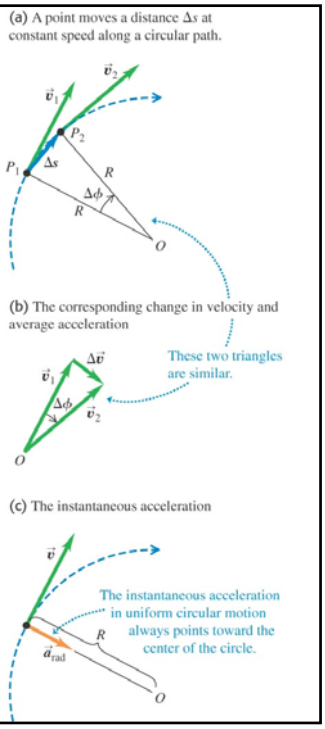
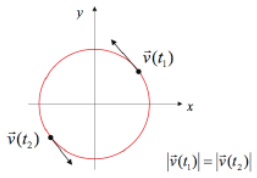
Finding motion information

- velocity change,
- average acceleration,
- and instantaneous acceleration

may be found.



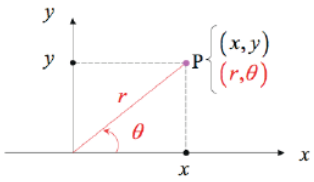
Uniform Circular Motion



Magnitude of the velocity remains constant,

but direction of the velocity changes continuously

$\Rightarrow$ motion is accelerated



P can be located independently by either

Cartesian coordinates (x, y) or polar coordinates (r, θ)

The easiest way is to use polar coordinates

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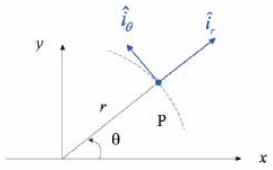
Unit vectors in polar coordinates



Introduce unit vectors $\hat{\imath}_r$ and $\hat{\imath}_\theta$:

$\hat{\imath}_r$ points in the direction of increasing r when θ is held constant,
$\hat{\imath}_\theta$ points in the direction of increasing θ when r is held constant

Recall that $\hat{\imath}$ and $\hat{\jmath}$ are constant in both magnitude and direction,

$\hat{\imath}_r$ and $\hat{\imath}_\theta$ are constant in magnitude, but they are not constant in direction.

As P moves with respect to time, these polar unit vectors change their directions

Consequently, they have non-zero time derivatives

We need to calculate those time derivatives

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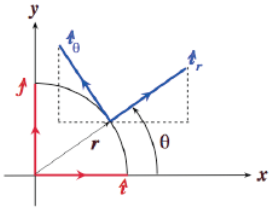
Time derivatives of unit vectors

Express polar unit vectors in terms of Cartesian unit vectors:



$$\begin{cases} \hat{\imath}_r &= \hat{\imath}\,\cos\theta &+ \hat{\jmath}\,\sin\theta \\ \hat{\imath}_\theta &= -\hat{\imath}\,\sin\theta &+ \hat{\jmath}\,\cos\theta \end{cases}$$

Differentiating radial unit vector:

$$\begin{aligned} \Rightarrow \frac{d}{dt}\hat{\imath}_r &= -\hat{\imath}\,\sin\theta\,\frac{d\theta}{dt} + \hat{\jmath}\,\cos\theta\,\frac{d\theta}{dt} \\ &= \frac{d\theta}{dt}\left(-\hat{\imath}\,\sin\theta + \hat{\jmath}\,\cos\theta\right) = \frac{d\theta}{dt}\hat{\imath}_\theta \equiv \omega\hat{\imath}_\theta \end{aligned}$$

where ω is called the angular velocity

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Time derivative of azimuthal unit vector

Differentiating azimuthal unit vector:

$$\begin{aligned} \Rightarrow \frac{d}{dt}\hat{\imath}_\theta &= -\hat{\imath}\,\cos\theta\,\frac{d\theta}{dt} - \hat{\jmath}\,\sin\theta\,\frac{d\theta}{dt} \\ &= -\frac{d\theta}{dt}\left(\hat{\imath}\,\cos\theta + \hat{\jmath}\,\sin\theta\right) = -\frac{d\theta}{dt}\hat{\imath}_r = -\omega\hat{\imath}_r \end{aligned}$$

Summary: $d\hat{\imath}_r/dt = \omega\hat{\imath}_\theta \;\;,\;\; d\hat{\imath}_\theta/dt = -\omega\hat{\imath}_r$

Differentiating position vector $\vec{r} = r\hat{\imath}_r$ gives needed kinematical relation for velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{\imath}_r + r\frac{d\hat{\imath}_r}{dt} = \frac{dr}{dt}\,\hat{\imath}_r + r\omega\,\hat{\imath}_\theta$$

Radial component of velocity is $v_r = dr/dt$

Tangential component of velocity is $v_\theta = \omega r$

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Kinematical relations for acceleration

Differentiating velocity gives

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v_r\,\hat{\imath}_r + \omega\, r\,\hat{\imath}_\theta\right) \\
&= \left(v_r\frac{d\hat{\imath}_r}{dt} + \hat{\imath}_r\frac{dv_r}{dt}\right) + \left(\omega\, r\,\frac{d\hat{\imath}_\theta}{dt} + \omega\,\hat{\imath}_\theta\frac{dr}{dt} + r\,\frac{d\omega}{dt}\,\hat{\imath}_\theta\right) \\
&= v_r\,\omega\hat{\imath}_\theta + \hat{\imath}_r\frac{dv_r}{dt} - \omega^2\, r\,\hat{\imath}_r + \omega\underbrace{\left(\frac{dr}{dt}\right)}_{v_r}\hat{\imath}_\theta + r\,\frac{d\omega}{dt}\,\hat{\imath}_\theta \\
&= \hat{\imath}_r\left(\frac{dv_r}{dt} - \omega^2\, r\right) + \hat{\imath}_\theta\left(2v_r\,\omega + r\,\frac{d\omega}{dt}\right)
\end{aligned}$$

General expression for acceleration is quite complicated

It is simpler for motion on a circular path $(v_r = 0)$ at constant speed $(d\omega/dt = 0)$

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Circular motion with constant speed

- For motion in a circle at constant speed, $r = \text{const}$ and $\omega = \text{const}$

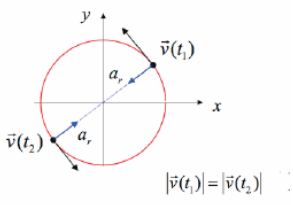
$$\Rightarrow \left\{ v_r = \frac{dr}{dt} = 0\ ,\ \frac{dv_r}{dt} = 0\ ,\ \frac{d\omega}{dt} = 0 \right\} \Rightarrow \vec{a} = -\omega^2 r\,\hat{\imath}_r$$

Now acceleration is totally in the radial direction and is oriented toward the center of motion

It is called centripetal acceleration.

An alternative way of expressing the centripetal acceleration:

$$a_r = -\omega^2 r = -\frac{v_\theta^2}{r^2}\,r = -\frac{v_\theta^2}{r}$$



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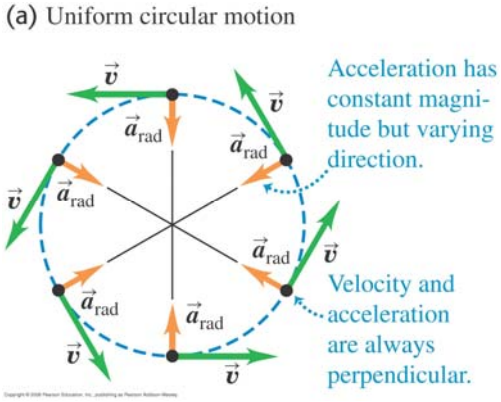
Circular motion with constant speed

$$a_{rad} = \frac{v^2}{r} = \omega^2 r$$

$$T = \frac{2\pi r}{v}$$



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Part 4

Relative Motion of Inertial Frames

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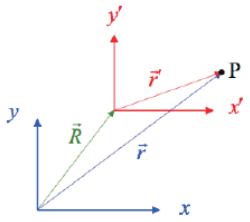
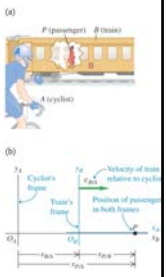
Kinematical relations for acceleration

Consider a frame (x', y'), with axes parallel to those of (x, y) frame, and the origin of (x', y') located at $\vec{R} = \vec{u}t$,

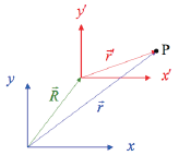




where $\vec{u}$ is the constant velocity of the primed frame relative to the unprimed frame

Such frames are called
inertial frames of reference

Relations between two inertial frames



Note that
$\vec{r} = \vec{R} + \vec{r}'$

Differentiate with respect to time to obtain
$\vec{v} = \vec{u} + \vec{v}'$

Since $\vec{u}$ is constant, one further time-differentiation yields
$\vec{a} = \vec{a}'$

This result will be particularly important when we will consider the principles of Newtonian dynamics