the end of the barrel, the monkey drops from the tree toward the ground. The bullet will:
A. pass over the monkey's head.
B. hit the monkey.
C. pass below the monkey.

5. A rider on a galloping horse holds a heavy ball out to his side and drops it. The ball will hit the ground:
A. immediately below the point where it was dropped.
B. immediately below the point the horse and rider are when the ball reaches the ground.
C. some position other than A or B. Please specify.

6. A boat is sailing rapidly along the surface of a smooth lake. A heavy ball is dropped from the top of a high mast. The ball will fall and
A. strike the deck at the foot of the mast.
B. strike the deck behind the foot of the mast depending on the speed of the boat.
C. strike the deck in front of the mast depending on the speed of the boat.

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Maximum projectile range with drag and lift, with particular application to golf

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This paper explores the interesting problem of projectile motion without the vacuum idealization. Particular attention is paid to golf ball trajectories with and without lift. No lift trajectories with linear and quadratic drag are considered first. Then, trajectories with lift and linear drag are investigated. Projection angles for maximum range are determined for all these cases. Computer solutions are used throughout, with a Runge–Kutta routine used for all cases except for the well-known closed solution for the no lift, linear drag projectile.

I. INTRODUCTION

One of the interesting questions which is usually left unanswered in introductory or intermediate mechanics courses[1] is "What angle do you need for maximum projectile range if you are not in vacuum?" Virtually everybody has had the experience of hitting a golf ball, or a baseball, or throwing a football, and there is a natural curiosity about how to project a ball in air in order to achieve maximum range.

Three previous papers in this Journal have addressed the problem of the projection angle for maximum range for the shot put,[2] the discus,[3] and for round pebbles or stones.[4] In the case of the shot put the air resistance has little effect on the range, whereas in our analyses the drag is always a significant factor. In the case of the discus analysis, the author assumes quadratic drag and lift forces, whereas our analysis of the golf ball problem uses linear drag and lift forces. The discus paper also has the effect of wind as a major factor, in our paper we assume no wind. It is interesting to note that the optimum projection angles for the quadratic drag golf ball with no lift found in this paper (38° and 35°) are similar to the optimum discus projection angles found by Frohlich[3] (33° to 39°) for no wind conditions. The paper on throwing pebbles assumes a quadratic drag force and no lift.

The author's interest in these matters was all the greater because he has been an avid golfer for a long time. The large discrepancy between the approximately 11 deg of loft for the golf driver club and the 45 deg maximum range angle for a vacuum was the motivation to begin a study of the question of maximum projectile range in the presence of air resistance, with particular application to the flight of a golf ball.

Of course, the first question which should be addressed is the nature of the resisting, or drag force. There was evidence in the literature[5] that the drag force on a golf ball is linear. This case is well known and can be solved exactly if there is no lift. However, the aerodynamic lift on a golf ball due to the backspin of the ball is a major force on the ball. Thus to do anything with the golf ball problem requires an analysis with three forces: gravity, drag, and lift. In this paper we start with the treatment of no lift projectiles, and then turn to the problem of trajectories with lift.

In 1949 Davies[6] investigated the lift and drag forces on spinning golf balls. He used the B. F. Goodrich wind tunnel with a wind stream velocity of 105 ft/s, and dropped rotating golf balls into this wind stream. The rotational speeds used were up to 8000 rpm. Davies found that the drag "increased nearly linearly from about 0.06 lb for no spin to about 0.1 lb at 8000 rpm" and that "the lift varied with the rotational speed as $L = 0.064[1 - \exp(-0.000\,26N)]$" ($L$ in lb and N in rpm).

In 1959 Williams[5] carried out an analysis which utilized data on golf ball carry as a function of projection velocity. The Reynolds number for Davies's work was 88 000; Williams's work was for the higher Reynolds numbers actually achieved in golf play. He used initial velocities in the range from 150 to 225 ft/s (Reynolds numbers from about 1.25×10^5 to 1.9×10^5). Williams showed that in this range of Reynolds numbers the drag force varied linearly with the velocity. The drag force is customarily written as

$$D = C_D A\,(1/2)\,\rho v^2\,, \tag{1}$$

where C_D is the drag coefficient, A is the cross-sectional area of the ball, ρ is the air density, and v is the ball speed. Williams showed that $C_D = (46/v)$ where v is in ft/s. Thus the drag coefficient drops significantly for higher speeds, giving the hard hitter an advantage.[7] A similar drop in the drag coefficient for smooth spheres occurs at a Reynolds number just above 2×10^5.[8] Since the drag coefficient for the golf ball goes inversely with v, Williams found that the drag force varied linearly with the speed, specifically that the drag force D was

$$D = 0.000\,783v \text{ lb}\,. \tag{2}$$

Williams's calculations were for the British ball whose diameter is 1.62 in., as compared to a diameter of 1.68 in. for the American ball.

II. LINEAR DRAG

As a first step, we consider the case of a nonspinning golf ball with linear air resistance. The component dynamical equations for this case are

$$-cv_x = m\frac{dv_x}{dt}\,, \tag{3}$$

$$-mg - cv_y = m\frac{dv_y}{dt} \tag{4}$$

with the well-known solutions

$$x = \frac{m}{c}v_{x_0}(1 - e^{-(c/m)t})\,, \tag{5}$$

$$y = \frac{m}{c}\left(\frac{mg}{c} + v_{y_0}\right)(1 - e^{-(c/m)t}) - \frac{mgt}{c}\,. \tag{6}$$

In this solution the golf ball starts at $x = y = 0$ with initial velocity components v_{x_0} and v_{y_0} at $t = 0$.

The weight of a golf ball is 1.62 oz and the projection speed of a good drive is about 200 ft/s.[9] The value of c, as given by Eq. (2) above, is $c = 0.000\,783$ lb/(ft/s). Hence,

$$\frac{c}{m} = \frac{0.000\,783}{(1.62/16)/32} = 0.25 \text{ s}^{-1}\,. \tag{7}$$

The terminal velocity of a freely falling golf ball is, therefore,

$$v_t = \frac{mg}{c} = \frac{32}{0.25} = 128 \text{ ft/s} = 87 \text{ mph}\,. \tag{8}$$

If we substitute the value for c/m given by Eq. (7) in Eqs. (5) and (6), and we use $v_0 = 200$ ft/s, we get

$$x = 800(\cos\theta_0)(1 - e^{-t/4})\,, \tag{9}$$

$$y = 4(128 + 200\sin\theta_0)(1 - e^{-t/4}) - 128t\,. \tag{10}$$

Figure 1 shows a plot of these coordinates for projection angle θ_0 in 5-deg intervals from 10° to 50°. Maximum range is achieved for an angle between 30° and 35°. Additional computer runs show that the maximum range of 503.9 ft occurs for $\theta_0 = 32°$.

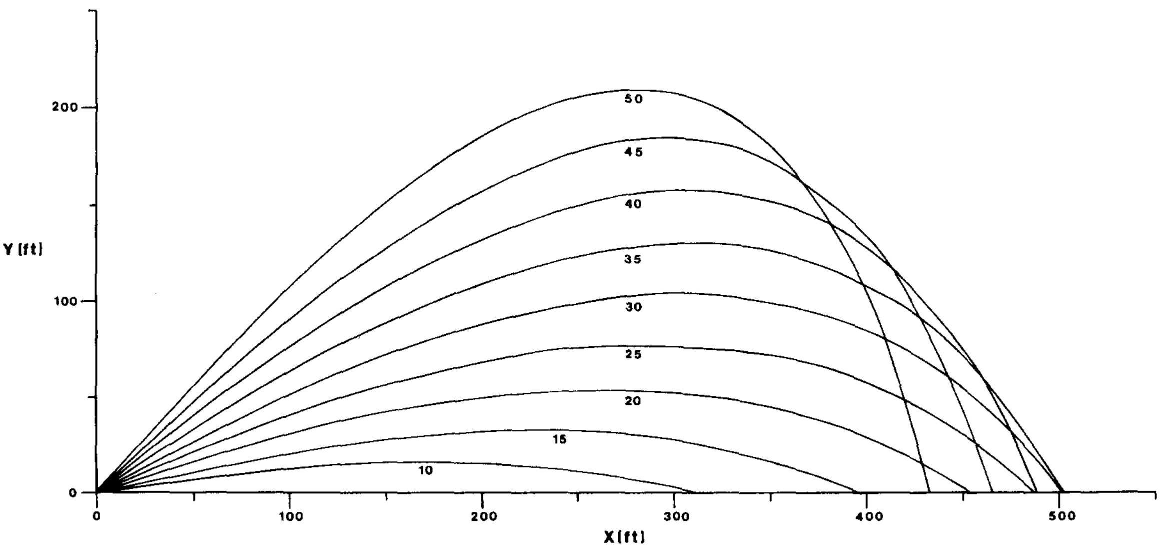

Fig. 1. Trajectories for various projection angles, linear drag, $c/m = 0.25$.

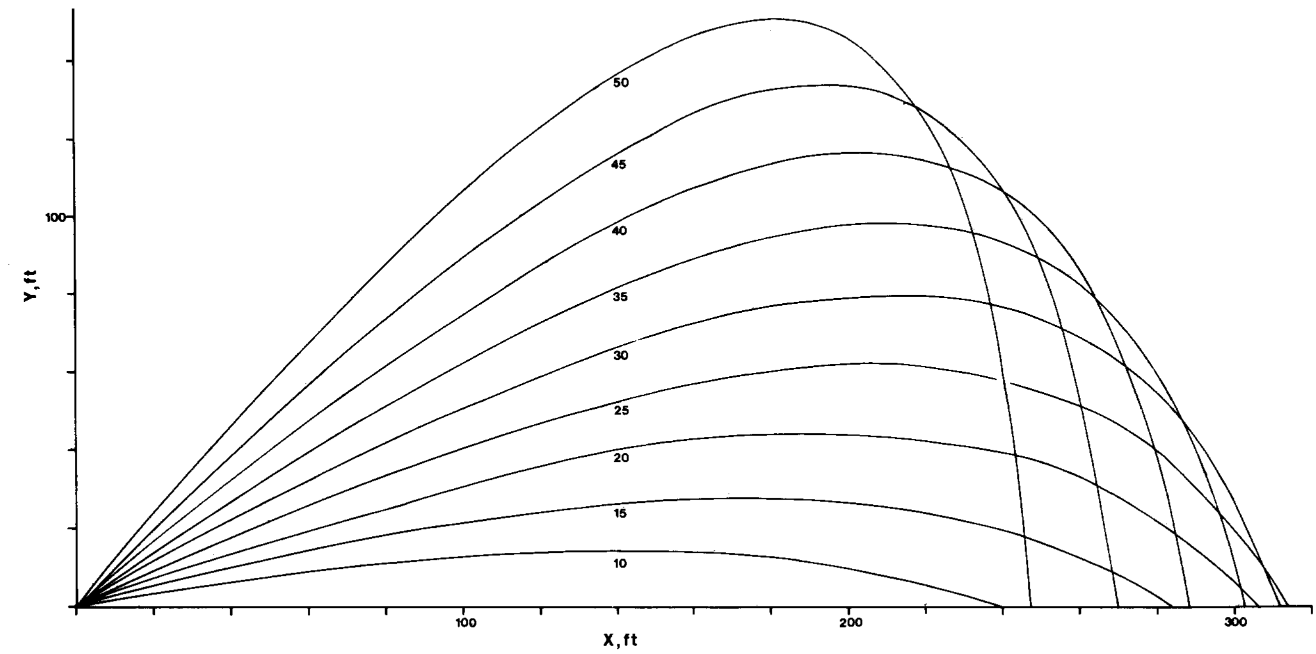

Fig. 2. Trajectories for various projection angles, linear drag, $c/m = 0.5$.

It is of interest to examine the effect of varying the drag coefficient c. Suppose, for example, that the drag is doubled so that $c/m = 0.5$. Under these circumstances we have

$$x = 400 \cos \theta_0 (1 - e^{-t/2}), \tag{11}$$

$$y = 2(64 + 200 \cos \theta_0)(1 - e^{-t/2}) - 64t. \tag{12}$$

Figure 2 shows a plot of Eqs. (11) and (12) for θ_0 in 5-deg intervals from 10° to 50°. The maximum range is now 314.2 ft, and it occurs for a projection angle of 26°.

III. QUADRATIC DRAG

For low speeds in air the drag force goes as the square of the speed. Although the well-driven golf ball faces linear drag, the terminal velocity of 128 ft/s found in Eq. (8) can be used to get a hypothetical quadratic drag coefficient for a slowly moving golf ball as follows:

$$c_2 v_t^2 = mg,$$

$$\frac{c_2}{m} = \frac{g}{v_t^2} = \frac{32}{(128)^2} = \frac{1}{512}\ \text{ft}^{-1}. \tag{13}$$ [10]

The equations of motion for the case of quadratic drag are

$$-c_2 v^2 \cos \theta = m \frac{dv_x}{dt}, \tag{14}$$

$$-mg - c_2 v^2 \sin \theta = m \frac{dv_y}{dt} \tag{15}$$

or

$$\frac{dv_x}{dt} = -\frac{c}{m}\sqrt{v_x^2 + v_y^2}\, v_x, \tag{16}$$

$$\frac{dv_y}{dt} = -g - \frac{c}{m}\sqrt{v_x^2 + v_y^2}\, v_y. \tag{17}$$

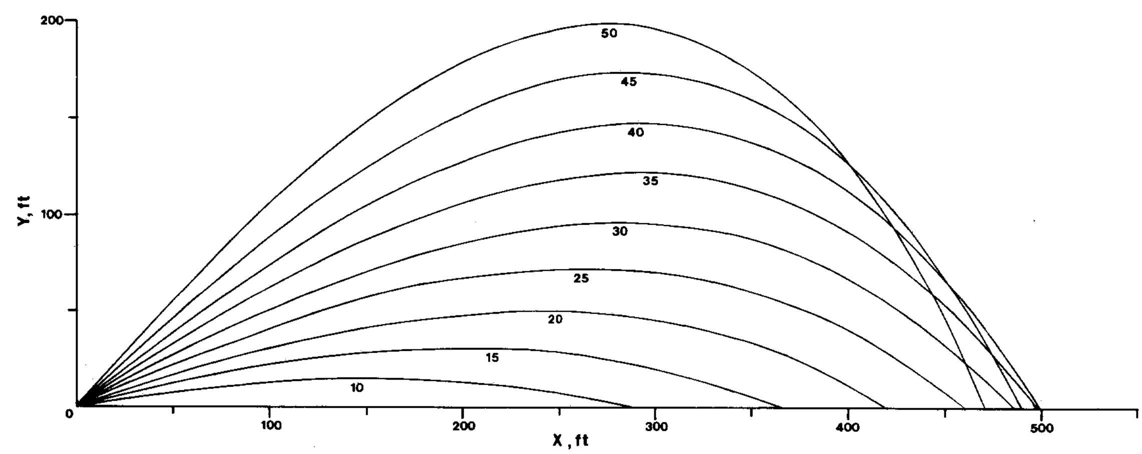

Fig. 3. Trajectories for various projection angles, quadratic drag, $c/m = 1/512$.

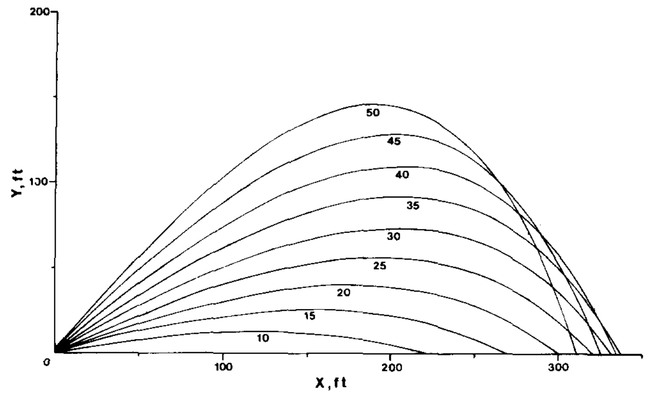

Fig. 4. Trajectories for various projection angles, quadratic drag, $c/m = 1/256$.

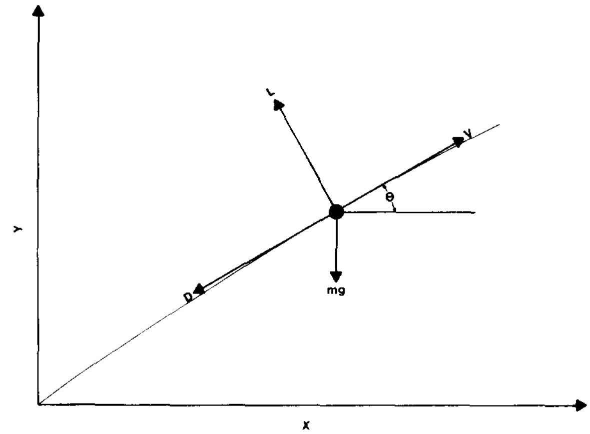

Fig. 5. Forces on a golf ball hit with backspin.

These coupled differential equations were solved numerically using the Runge–Kutta method. The subroutine RKGS from the IBM System/360 Scientific Subroutine Package, Version III, was used for this purpose.

Figure 3 shows the Runge–Kutta solution. A maximum range of 499.1 ft was achieved at an initial projection angle of 38 deg. The drag coefficient was then doubled so that

$$\frac{c_2}{m} = \frac{1}{256}\ \text{ft}^{-1} . \tag{18}$$

Figure 4 shows the Runge–Kutta solution for this case. Now the maximum range was 336.6 ft for a projection angle of 35 deg.

IV. CHECK OF RUNGE–KUTTA ROUTINE

The Runge–Kutta routine RKGS was submitted to an accuracy check by comparing its solution for the linear drag case with the exact solution. The Runge–Kutta solution agreed to at least four places with the exact solution.

V. GOLF BALL WITH LIFT AND LINEAR DRAG

The forces on a golf ball hit with backspin are shown in Fig. 5. The lift force can be written as

$$L = C_L A \tfrac{1}{2} \rho v^2 . \tag{19}$$

This lift force is due to a pressure difference caused by the spinning ball. The backspin of the ball increases the air speed over the top of the ball and decreases it below the ball. Thus the air pressure is decreased above the ball and increased below the ball, providing an aerodynamic lift force. As a first approximation, it seems reasonable to assume that this varies linearly with ω, the backspin angular speed, and v, the speed of the ball. The linear dependence on ω is essentially in agreement with the experimental work of Davies quoted earlier. We also assume that C_L varies inversely with v, just as Williams found for C_D. Since ω does not change appreciably during the flight of the ball,[13] we will take it to be constant. This gives us a lift force which varies linearly with the speed v,

$$L = kv . \tag{20}$$

Newton's equations in the x and y directions are

$$-cv_x - kv \sin\theta = m \frac{dv_x}{dt} ,$$

$$-\frac{c}{m} v_x - \frac{k}{m} v_y = \frac{dv_x}{dt} , \tag{21}$$

$$-cv_y + kv \cos\theta - mg = m \frac{dv_y}{dt} ,$$

$$-\frac{c}{m} v_y + \frac{k}{m} v_x - g = \frac{dv_y}{dt} . \tag{22}$$

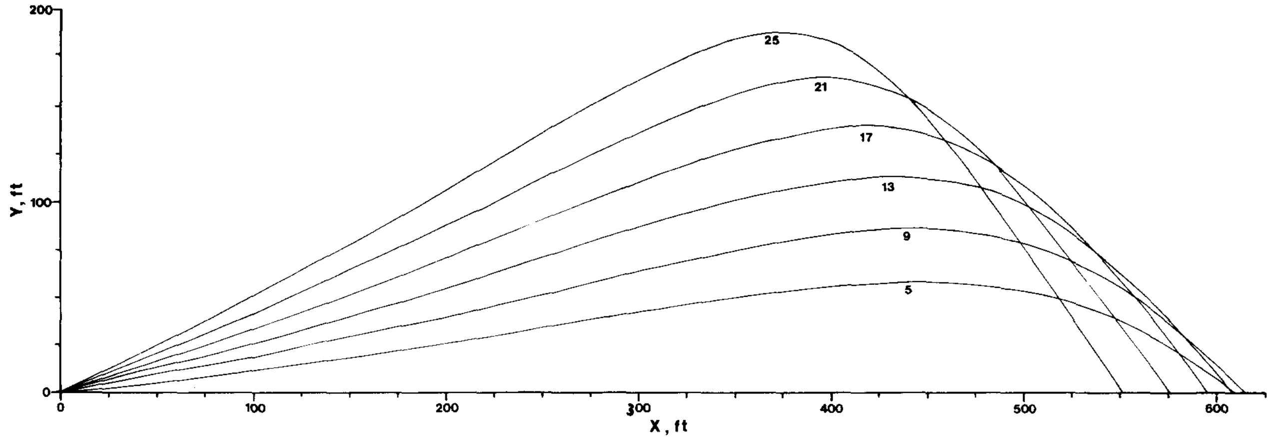

Fig. 6. Golf ball trajectories for various projection angles, $c/m = 0.25$, $k/m = 0.247$.

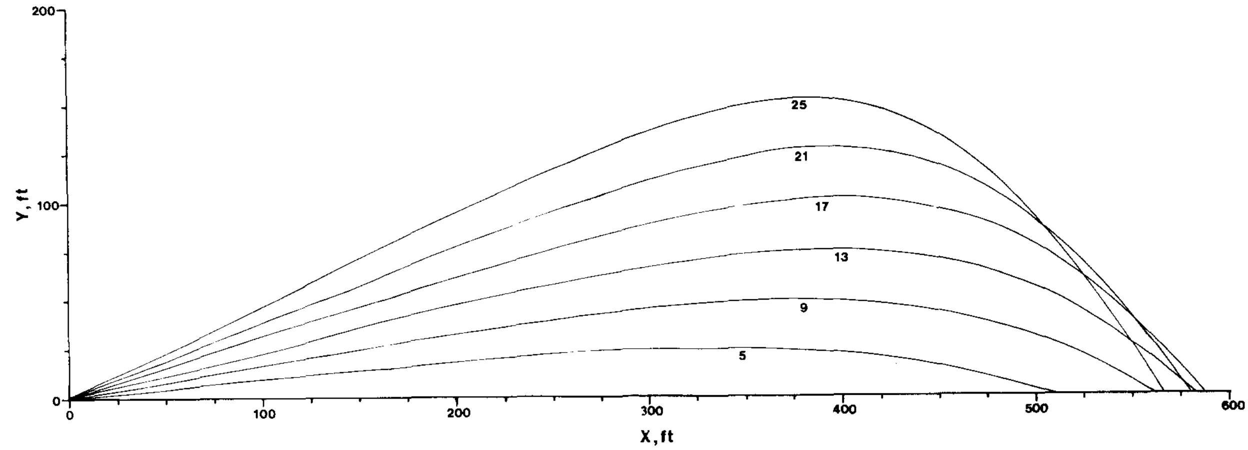

Fig. 7. Golf ball trajectories for various projection angles, $c/m = 0.25$, $k/m = 0.179$.

The initial lift on the golf ball is about 2.5 oz.[13] Hence, k/m can be calculated as follows:

$$kv_0 = 2.5/16 \text{ lb},$$

$$\frac{kv_0}{m} = \left(\frac{2.5}{16}\right) \left[\left(\frac{1.62}{16}\right)/32\right],$$

$$\frac{k}{m} = \frac{2.5(32)}{1.62(200)} = 0.247 \text{ s}^{-1}. \quad (23)$$

We once again take $c/m = 0.25$, as in Eq. (7). The Runge–Kutta solution of Eqs. (21) and (22) with these values for k/m and c/m is shown in Fig. 6. The maximum range of 614.6 ft is obtained for $\theta_0 = 9°$.

The values used above for lift and drag coefficients are very nearly the same. On the other hand, Cochran and Stobbs say: "For instance, as the ball flies away from the tee, drag may be at three to four ounces and lift at about two and a half ounces."[14] If we take the ratio of lift to drag coefficients to be 2.5/3.5 we obtain

$$\frac{k}{m} = \left(\frac{2.5}{3.5}\right)(0.25) = 0.179 \text{ s}^{-1}. \quad (24)$$

The computer solution of Eqs. (21) and (22) with $k/m = 0.179$ and $c/m = 0.25$ is shown in Fig. 7. The maximum range of 587.6 ft is now achieved for $\theta_0 = 16°$. This compares with Cochran and Stobb's investigation which found maximum carry at $\theta_0 = 20°$.[15] These last results seem to call for club manufacturers to fashion drivers at more than the approximate 10° that is common. However, Cochran pointed out that the use of a more lofted driver would reduce ball projection velocity and increase backspin, with an overall reduction in range.[16] Moreover, although the normal driver swung to catch the ball at a higher projection angle will maximize carry, it may not maximize total length since the ball will now hit the ground at a steeper angle. Cochran and Stobbs conclude:

> In most combinations of conditions, the longest drive will often be the one that goes off at 12° or 13°—a little higher than the starting angle of most people's drives, most of the time, but not very much so.
>
> In general, in fact, comparatively little may be gained in total distance off the tee even by the most strenuous and drastic attempts to alter the trajectory. For the vast majority of golfers, hitting the ball true and in good timing will prove much more important to good driving than "flying this one a bit higher."[17]

ACKNOWLEDGMENTS

The author acknowledges the assistance of Alan Benimoff of the College of Staten Island in the drawing of the figures. The kind help of Ed Van Dyke, Director, R & D and Quality Control of the Ben Hogan Company, in providing reference material is also acknowledged.

[1]A typical case is K. R. Symon, *Mechanics* (Addison-Wesley, Reading, MA, 1971), which treats the case of linear drag but does not consider the problem of maximum range, except for an unexplained Fig. 3.29 (p. 115), which shows "trajectories for maximum range for projectiles with various muzzle velocities."

[2]D. B. Lichtenberg and J. G. Wills, Am. J. Phys. **46**, 546 (1978).

[3]C. Frohlich, Am. J. Phys. **49**, 1125 (1981).

[4]J. A. Zufiria and J. R. Sanmartín, Am. J. Phys. **50**, 59 (1982).

[5](a) D. Williams, Quart. J. Mech. Appl. Math. **XII** (3), 387–392 (1959). However, (b) A. Cochran and J. Stobbs in *The Search for the Perfect Swing* (Lippincott, Philadelphia and New York, 1968), p. 160 indicate that the drag increases a little faster than the speed. They say "as a ball's speed doubles from, say 100 to 200 feet per second, the drag generated goes up about two and a half times." This would give a drag force which goes as $v^{1.3}$. *The Search for the Perfect Swing* presents the results of a team of researchers in Great Britain who conducted a scientific study of various aspects of golf. This investigation was sponsored by the Golf Society of Great Britain. The book was written for a popular audience and therefore does not contain many of the research details which would be presented in journal papers. A further volume written for scientists is promised (p. 227) but has not yet appeared to this author's knowledge.

[6]J. M. Davies, J. Appl. Phys. **20**, 821 (1949).

[7]This advantage comes from the horizontal distance covered by the hard hitter's golf ball during the time its velocity is falling to the lower one of the average hitter. If the drag coefficient were to remain constant instead of dropping with v, Williams calculated that 43 yards would be covered by the ball in going from 225 to 190 ft/s. The experimental difference in carry for these two projection velocities was found to be 52 yards. Thus the hard hitter has "gained" 52 − 43 = 9 yards, due to the drop in the drag coefficient. Williams concluded "It can be said therefore that only some 17 per cent of the long hitter's advantage in distance can be attributed to the drop in drag coefficient. The rest apparently he gains by honest effort" (p. 392).

[8]See, for example, V. L. Streeter and E. B. Wylie, *Fluid Mechanics*, 6th ed. (McGraw-Hill, New York, 1975), Fig. 5.21 on p. 279.

[9]Reference 5(b), p. 163.

[10]The drag coefficient C_D corresponding to this value of c_2/m for the

British ball can be readily determined:

$$\frac{c_2}{m} = \left(\frac{C_D}{m}\right) A \tfrac{1}{2}\rho, \quad A = 0.0143\ \text{ft}^2,$$
$$\rho = 0.002\,37\ \text{slugs/ft}^3, \quad m = (1.62/16)/32\ \text{slugs},$$
$$C_D = 0.36.$$

[11]Reference 5(b), p. 161.
[12]Reference 5(b), p. 160.
[13]Reference 5(b), p. 164.
[14]Reference 5(b), pp. 164–165.
[15]Reference 5(b), p. 165.

Latent heat and low-temperature heat capacity experiment for the general physics laboratory

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An experiment to measure the latent heat of vaporization of liquid nitrogen and the average heat capacities at constant pressure of several materials in the temperature range 77–295 K is described. This exercise uses normally available laboratory apparatus and is suitable for the general physics laboratory.

I. INTRODUCTION

We have recently developed a laboratory experiment for our general physics courses that incorporates the use of a cryogenic fluid (liquid nitrogen—hereafter called LN_2), the measurement of its latent heat of vaporization, and the study of the (average) low-temperature heat capacities of several materials and their relation to the law of Dulong and Petit. The experiment is simple and uses normally available laboratory apparatus (assuming a source of LN_2 is available). Data collection and analysis can be performed in a two-hour laboratory period. The values obtained have small experimental uncertainties and agree well with values calculated from published data.

II. LATENT HEAT OF VAPORIZATION OF LIQUID NITROGEN

A schematic of the apparatus used to measure the latent heat of vaporization of LN_2 is shown in Fig. 1. An electrical heater R was suspended in a LN_2 bath in a cup assembly which rested on a scale balance pan. As the LN_2 boiled away a background loss rate was determined by measuring $m(t)$, the mass of the LN_2 plus cup assembly, as a function of time. After a few minutes of observation, switch S was closed. The current I and voltage V were measured. Simultaneously, measurements of $m(t)$ continued but the rate was, of course, much faster. A good technique to measure $m(t)$ is to unbalance the scale one or two grams too light and read the time as the pointer passes zero, then unbalance again and repeat. A suitably damped dial-type balance makes this process rapid and simple.

After a few grams of LN_2 had been boiled off by the heater, switch S was opened again and the heater time interval, Δt, carefully noted. (A separate timer for this purpose is advantageous). Measurements of $m(t)$ were continued for a few minutes to re-establish the background. These data were plotted as shown in Fig. 2.

The double-wall styrofoam cup assembly, labeled item $\mathscr{C}$ in Fig. 1, was made by suspending a 6-oz cup inside a 14-oz cup. A styrofoam ring was used to center the cups and a bead of silicone bathtup-type sealant sealed the joint and stabilized the assembly. The analytical balance[1] had a 0–10 g dial. The heater probe had a 33-Ω, 10-W wire wound resistor (nominal value) mounted on the end of a small stainless steel tube. The power source was a 28-V (nominal) laboratory supply. The 10-W rating of $\mathscr{R}$ was exceeded but the LN_2 cooling compensated. The current and voltage were measured using Keithley Model 130 digital multimeters.[2] Values obtained for the data shown in Fig. 2 were $V = 28.5$ V and $I = 0.871$ A.

Straight lines were drawn through the data of Fig. 2. From the vertical displacement of the two lines near the center of the heating period the mass of the LN_2 boiled off was determined to be 9.35 g. The heating interval was timed to be 73.4 s. The latent heat of vaporization L_v was calculated from the relation

$$L_v = VI\Delta t/\Delta M. \tag{1}$$

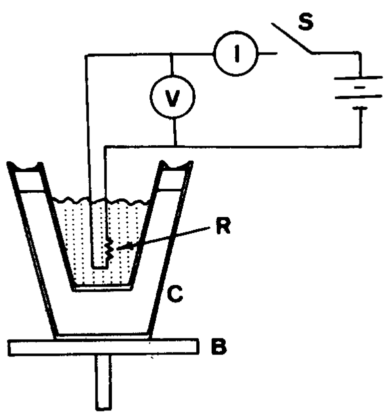

Fig. 1. Schematic of the apparatus used to measure the latent heat of vaporization of LN_2. The letters refer to the following items: B, balance; C, styrofoam cup assembly; I, current meter; R, 33 Ω resistor; S, switch; V, voltmeter. The power source was 28 V.