Physics 313 Astrophysics Spring 2009
Prof C. Hyde chyde'@'odu.edu
www.odu.edu/~chyde/Teaching
Homework 7 due Wed March 25, 2009

1) The speed of light in a medium of index of refraction n is c/n, where c is the speed of light in vacuum. Relativistic charged particles can travel in the medium with a velocity v such that $c/n < v < c$. In this case, the particle will emit visible light all along its trajectory. The light is emitted at an angle θ relative to the particle direction, with $\cos(\theta) = c/(nv)$. The index of refraction of water is 1.33 .

   The relativistic energy E of a particle of rest mass m is
   $E=mc^2\gamma$ $\gamma = [1-\beta^2]^{-1/2}$ $\beta=v/c$.

   a. The rest mass of the electron is mc^2=0.511 MeV. Find the β value (close to 1) of an electron of energy 5.11 MeV.
      $\gamma = E/mc^2$ = (5.11 MeV) / (0.511 MeV) = 10;
      $\beta = [1-1/\gamma^2]^{1/2} = [1-0.01]^{1/2} = [0.99]^{1/2} = 0.995$
   b. Find the Cerenkov angle for this electron in water.
      $\cos(\theta) = c/(nv) = 1/(nb) = 1/(0.995*1.33) = 0.756.$
      θ = 40.9° .
   c. What is the lowest energy electron that will produce Cerenkov light in water (this limit is β=1/n since the cosine function cannot be >1).
      $E=mc^2\gamma = mc^2 / [1-\beta^2]^{1/2} = mc^2 / [1-1/n^2]^{1/2}$
      E = (0.511 MeV) / $[1-1/1.33^2]^{1/2}$ = (0.511 MeV) / $[0.435]^{1/2}$
      E = 0.775 MeV
      Kinetic Energy = E−mc^2 = 0.264 MeV.

2) An ultra-high energy cosmic ray proton can be slowed down by the following inelastic collision with the photons of the 3° K cosmic black body radiation (CBR)

$\gamma+p \rightarrow \Delta \rightarrow p+\pi^0$.

The photons in the CBR have a typical energy of $2.5 \bullet 10^{-4}$ eV. The proton has a rest mass of $Mc^2=938 \bullet 10^6$ eV and the Δ-particle has a mass $M_\Delta c^2=1232 \bullet 10^6$ eV. Consider just the head on collision $\gamma+p \rightarrow \Delta$. Energy and momentum must be conserved in this reaction.

Energy Conservation $k+E=E_\Delta$.

Momentum Conservation $Pc-k=P_\Delta c$,

where k is the photon energy (and momentum times c), E is the proton energy, P is the proton momentum, E_Δ is the Delta energy and P_Δ is the Δ momentum. The following relativistic relation holds for any particle of mass m:

$E^2 = (pc)^2 + (mc^2)^2$.

a. Using the energy and momentum conservation equations, as well as the energy-momentum-mass relation for the proton and Δ, find the minimum proton energy E such that the reaction $\gamma+p \rightarrow \Delta$ is allowed.

Eliminate ED and PD by squaring the two equations and subtracting:

$(k+E)^2 = E_\Delta^2$

$(Pc-k)^2 = (P_\Delta c)^2$

$k^2+2kE+E^2 - [(Pc)^2-2Pck+k^2] = E_\Delta^2 - (P_\Delta c)^2 = (M_\Delta c^2)^2$

$2k[E+Pc] + E^2 - (Pc)^2 = (M_\Delta c^2)^2$;

$2k[E+Pc] = (M_\Delta c^2)^2 - (Mc^2)^2$

i. Approximate Solution, E>>Mc2, therefore $Pc \approx E$

$4kE \approx [(M_\Delta c^2)^2 - (Mc^2)^2]$

$E \approx [(M_\Delta c^2)^2 - (Mc^2)^2] / (4k)$

$E \approx [(1232 \bullet 10^6 \text{ eV})^2 - (938\ 10^6 \text{ eV})^2]/(0.001\text{eV})$

$= 6.4 \bullet 10^{20}$ eV = 102 J

The approximation E>>938 MeV is well justified

ii. Exact solution

$E+Pc = [(M_\Delta c^2)^2 – (Mc^2)^2] / (2k)$

$E+[E^2-(Mc^2)^2]^{1/2} = [(M_\Delta c^2)^2 – (Mc^2)^2] / (2k)$

Isolate the square root

$[E^2-(Mc^2)^2]^{1/2} = [(M_\Delta c^2)^2 – (Mc^2)^2] / (2k) – E$

$$E^2-(Mc^2)^2 = [(M_\Delta c^2)^2–(Mc^2)^2]^2/(2k)^2 – 2E[(M_\Delta c^2)^2– (Mc^2)^2] / (2k) – E^2.$$

$$2E[(M_\Delta c^2)^2– (Mc^2)^2] / (2k) = [(M_\Delta c^2)^2–(Mc^2)^2]^2/(2k)^2 + (Mc^2)^2$$

$$E = [(M_\Delta c^2)^2–(Mc^2)^2]/(4k) + k(Mc^2)^2/ [(M_\Delta c^2)^2– (Mc^2)^2]$$

The second term is a correction to the answer in i.

$E = 6.4•10^{20}$ eV + ($2.5•10^{-4}$ eV)1.4

The second term is utterly negligible, it is 24 orders of magnitude less than the first term