the excited spectrum of QCD
the spectrum of excited hadrons

let’s begin with a convenient fiction:

imagine that QCD were such that there was a spectrum of stable excited hadrons

e.g. suppose we set up QCD with just two degenerate flavours of quark with mass roughly that of the charm quark

\[ m_c = m_k \sim 1.5 \text{ GeV} \]

then we’d expect a spectrum of \( c\bar{k} \) hadron states starting at about 3 GeV that are stable up to about 6 GeV

except perhaps if glueballs are important?
the spectrum of excited hadrons

might expect something like the non-relativistic quark model

\[ c \bar{k} \left[ n \ 2S+1 \ L_J \right] \]

... but QCD might be more interesting than this, e.g. what about ‘gluonic excitations’?

- glueballs
- hybrids

... and we need to verify if our simple expectations of a $q\bar{q}$ spectrum are really present
we’d like to map out the spectrum of states in each $J^{PC}$

need interpolating fields that transform like the desired $J^{PC}$

$$\bar{\psi}\gamma_5\psi \sim 0^{-+}$$
$$\bar{\psi}\psi \sim 0^{++}$$
$$\bar{\psi}\gamma_i\psi \sim 1^{--}$$
$$\bar{\psi}\gamma_5\gamma_i\psi \sim 1^{++}$$
$$\epsilon_{ijk}\bar{\psi}\gamma_j\gamma_k\psi \sim 1^{+-}$$

... very limited in $J^{PC}$ coverage

one possible extension: include gauge-covariant derivatives

$$\mathbf{\overleftarrow{D}}_i = \mathbf{\overleftarrow{D}}_i - \mathbf{\overrightarrow{D}}_i = \mathbf{\overleftarrow{\partial}}_i - \mathbf{\overrightarrow{\partial}}_i - 2igA_i$$

e.g. $$\bar{\psi}\mathbf{\overleftarrow{D}}_i\psi \sim 1^{--}$$
$$\bar{\psi}\gamma_i\mathbf{\overleftarrow{D}}_j\psi \sim ?$$
$$i = 1 \ldots 3$$
$$j = 1 \ldots 3$$

9 elements

operator is reducible
the spectrum of excited hadrons

\[ \bar{\psi} \gamma_i \overleftrightarrow{D}_j \psi \sim ? \]

\( i = 1 \ldots 3 \)

\( j = 1 \ldots 3 \)

9 elements

operator is reducible

very easy to build a scheme where the operators are irreducible:

\[ \gamma_m \equiv \sum_i \epsilon_i(m) \gamma_i \quad \epsilon(m = \pm) = \mp \frac{1}{\sqrt{2}} [1, \pm i, 0] \]

\[ \overleftrightarrow{D}_m \equiv \sum_i \epsilon_i(m) \overleftrightarrow{D}_i \quad \epsilon(m = 0) = [0, 0, 1] \]

\[ \implies \langle 1m_1; 1m_2 | JM \rangle \bar{\psi} \gamma_{m_1} \overleftrightarrow{D}_{m_2} \psi \sim J^{++} \quad \text{with } J=0,1,2 \]

Hadron Spectrum Collaboration has used up to three derivatives:

\[ \langle 1m_1; j_{234}m_{234} | JM \rangle \]

\[ \langle 1m_3; j_{24}m_{24} | j_{234}m_{234} \rangle \]

\[ \langle 1m_2; 1m_4 | j_{24}m_{24} \rangle \]

\[ \bar{\psi} \gamma_{m_1} \overleftrightarrow{D}_{m_2} \overleftrightarrow{D}_{m_3} \overleftrightarrow{D}_{m_4} \psi \]

can build a big basis this way covering all \( J \leq 4 \)
the spectrum of excited hadrons

so we could compute correlators for each $J^{PC}$ and look at effective masses at large $t$

would give us the lightest state in each $J^{PC}$

we want more than this ...
the spectrum of excited hadrons

we need to be able to extract excited states

\[ C'(t) = \sum_n A_n e^{-E_n t} \]

a weighted sum of exponentials
- just do a fit to the time-dependence?

(fit variables: \(A_0, A_1 \ldots, E_0, E_1 \ldots\))

this is a very bad way to approach this problem

suppose two states are (nearly) degenerate
- fit won’t be able to tell if there are two states or one!

how do we determine how many states to include in the fit
- if we decrease \(t_{\text{min}}\) to use more of the data, need more states?

fortunately there is a very powerful method available ...
variational approach

suppose we have multiple operators for a given $J^{PC}$

\[ \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \ldots \]

\[ \psi \gamma_m \psi \]

\[ \psi \mathcal{D}_m \psi \]

\[ \langle 1m_1; 1m_2 | \mathcal{D}_m \psi \rangle \]

\[ \langle 1m_1; 2m_D | 1m \rangle \langle 1m_2; 1m_3 | 2m_D \rangle \psi \gamma_m \mathcal{D}_m \mathcal{D}_m \psi \]

\[
\begin{align*}
\text{e.g. } J^{PC} &= 1^{--} \\
C_{ij}(t) &= \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j(0) | 0 \rangle \\
&= \sum_n Z_i^{(n)} Z_j^{(n)} e^{-E_n t} \\
Z_i^{(n)} &= \langle n | \mathcal{O}_i(0) | 0 \rangle
\end{align*}
\]

solve the ‘generalised eigenvalue problem’ :

\[ C(t) \nu^{(n)} = \lambda_n(t) C(t_0) \nu^{(n)} \]

eigenvalues, ‘principal correlators’

\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

eigenvectors are ‘orthogonal’

\[ \nu^{(m)} \dagger C(t_0) \nu^{(n)} = \delta_{m,n} \]
the interpretation is relatively simple

the eigenvectors indicate the optimal linear combination of $O_i$ to interpolate $|n\rangle$

$$\Omega_n = \sum_i \nu_i^{(n)} O_i$$

$$\langle m | \Omega_n | 0 \rangle \approx \delta_{m,n}$$

degenerate states are easy to deal with - they might have $E_m = E_n$

- but they have orthogonal $\nu^{(m)}$, $\nu^{(n)}$
variational approach

principal correlators $t_0 = 15$
26 operators

variational analysis of 26×26 matrix of correlators

multiple approximate degeneracies

a E

0.4

0.5

0.6

0.7

0.8

superimposed J=1,3,4 spectra

' statistical' uncertainty (finite Monte Carlo sample)
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

Hadron Spectrum Collaboration
arXiv:1204.5425
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

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the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

if we’re interested in phenomenology, there is more information than what’s presented here

the relative sizes of \( \langle n | \mathcal{O}_i(0) | 0 \rangle \)

might tell us about the state composition?
back to the operators . . .

e.g. $J^{PC}=1^{-}$

consider a model-interpretation

$$\bar{\psi} \gamma_m \frac{1}{2}(1 - \gamma_0) \psi$$

spin-structure:

$$\psi \sim \left[ \frac{1}{\vec{\sigma} \cdot \vec{p}} \right] \chi$$

$$\frac{1}{2}(1 - \gamma_0) \psi \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi$$

upper component projector

$$\bar{\psi} \gamma_m \frac{1}{2}(1 - \gamma_0) \psi \sim \phi^\dagger \sigma_m \chi$$

$^{3S_1}$
back to the operators ... 

e.g. $J^{PC}=1^{--}$ consider a model-interpretation

$$\langle 1m_1; 2m_2 | 1m \rangle \bar{\psi} \gamma_{m_1} D^{[2]}_{J=2,m_2} \frac{1}{2} (1 - \gamma_0) \psi$$

$$D^{[2]}_{J,m} \equiv \langle 1m_1; 1m_2 | Jm \rangle \hat{D}_{m_1} \hat{D}_{m_2}$$

without gauge-fields: $D^{[2]}_{J=2,m} \rightarrow Y_2^m (\vec{\partial})$

$$\sim \langle 1m_1; 2m_2 | 1m \rangle \cdot \phi^{\dagger} \sigma m_1 \chi \cdot Y_2^{m_2} (\vec{q})$$

$q\bar{q}$ relative momentum
back to the operators . . .

e.g. $J^{PC}=1^{--}$

$$\bar{\psi} \gamma_5 D^{[2]}_{J=1,m} \frac{1}{2} (1 - \gamma_0) \psi$$

$$D^{[2]}_{J=1,m} \equiv \langle 1m_1; 1m_2|1m \rangle \overleftrightarrow{D}_{m_1} \overleftrightarrow{D}_{m_2}$$

without gauge-fields: $D^{[2]}_{J=1,m} \to 0$

with gauge-fields $D^{[2]}_{J=1,m} \propto [D_i, D_j] \propto F_{ij}$ chromomagnetic part of field-strength tensor

$$\bar{\psi} \gamma_5 t^a \psi \ B^a_m \quad \text{hyb}_1$$

$q \bar{q} (1S_0)$
operator overlaps

e.g. $J^{PC}=1^{--}$
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

"excess" $0^{--}, 1^{--}, 2^{--}$

Hadron Spectrum Collaboration
arXiv:1204.5425

$\text{JLab Advanced Study Institute}$
The charmonium spectrum from a lattice QCD calc can isolate dominant hybrid character across the spectrum.
hybrid mesons

a phenomenology of hybrid mesons based upon QCD calculations

a chromomagnetic field configuration is lowest excitation

\[ q\bar{q}_8(1S_0)B_8 \sim 0^{-+} \otimes 1^{+-} = 1^{--} \]
\[ q\bar{q}_8(3S_1)B_8 \sim 1^{--} \otimes 1^{+-} = (0, 1, 2)^{--} \]

\[ q\bar{q}_8(1P_1)B_8 \sim 1^{+-} \otimes 1^{+-} = (0, 1, 2)^{++} \]
\[ q\bar{q}_8(3P_0)B_8 \sim 0^{++} \otimes 1^{+-} = 1^{-+} \]
\[ q\bar{q}_8(3P_1)B_8 \sim 1^{++} \otimes 1^{+-} = (0, 1, 2)^{+-} \]
\[ q\bar{q}_8(3P_2)B_8 \sim 2^{++} \otimes 1^{+-} = (1, 2, 3)^{+-} \]
lighter quarks - isovector mesons

three flavours of quark - all at the strange quark mass

\[ m(\pi) \sim 700 \text{ MeV} \]
lighter quarks - isovector mesons

three flavours of quark - all at the strange quark mass

interpretations based on operator overlaps

\[ m(\pi) \sim 700 \text{ MeV} \]
lighter quarks - isovector mesons

three flavours of quark
- degenerate up/down quarks
- correct strange quark mass

\[ m(\pi) \sim 400 \text{ MeV} \]

\[ m_\pi = 396 \text{ MeV} \]

Hadron Spectrum Collab.
Phys.Rev.D82, 034508
isoscalar mesons

difference w.r.t. isovector mesons is addition of ‘disconnected’ diagrams

\[
\left[ \bar{\psi} \Gamma' \psi(t) \cdot \bar{\psi} \Gamma \psi(0) \right] - \left[ \bar{\psi} \Gamma' \psi(t) \cdot \bar{\psi} \Gamma \psi(0) \right]
\]

\[
\text{tr} \left[ Q_{t,0}^{-1} \Gamma Q_{0,t}^{-1} \Gamma' \right] - \text{tr} \left[ Q_{t,t}^{-1} \Gamma' \right] \text{tr} \left[ Q_{0,0}^{-1} \Gamma \right]
\]

challenging using ‘traditional’ methods
isoscalar mesons

hidden ‘light’ and hidden ‘strange’ can mix

\[ \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d}) \quad s\bar{s} \]

\[-2\]

\[-\sqrt{2}\]

\[-\sqrt{2}\]
**isoscalar mesons**

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PRD83 111502 (2011)

\[
m_\pi = 396 \text{ MeV}
\]

- **isoscalar**
  - $\ell_s$

- **isovector**
  - $\ell$

- **YM glueball**

0++ is a challenge
baryons

analogous large basis of operators for baryons - three quark fields respecting permutation (anti-)symmetry

Hadron Spectrum Collaboration
PRD84 074508 (2011)
PRD85 054016 (2012)

$m_\pi = 524 \text{ MeV}$