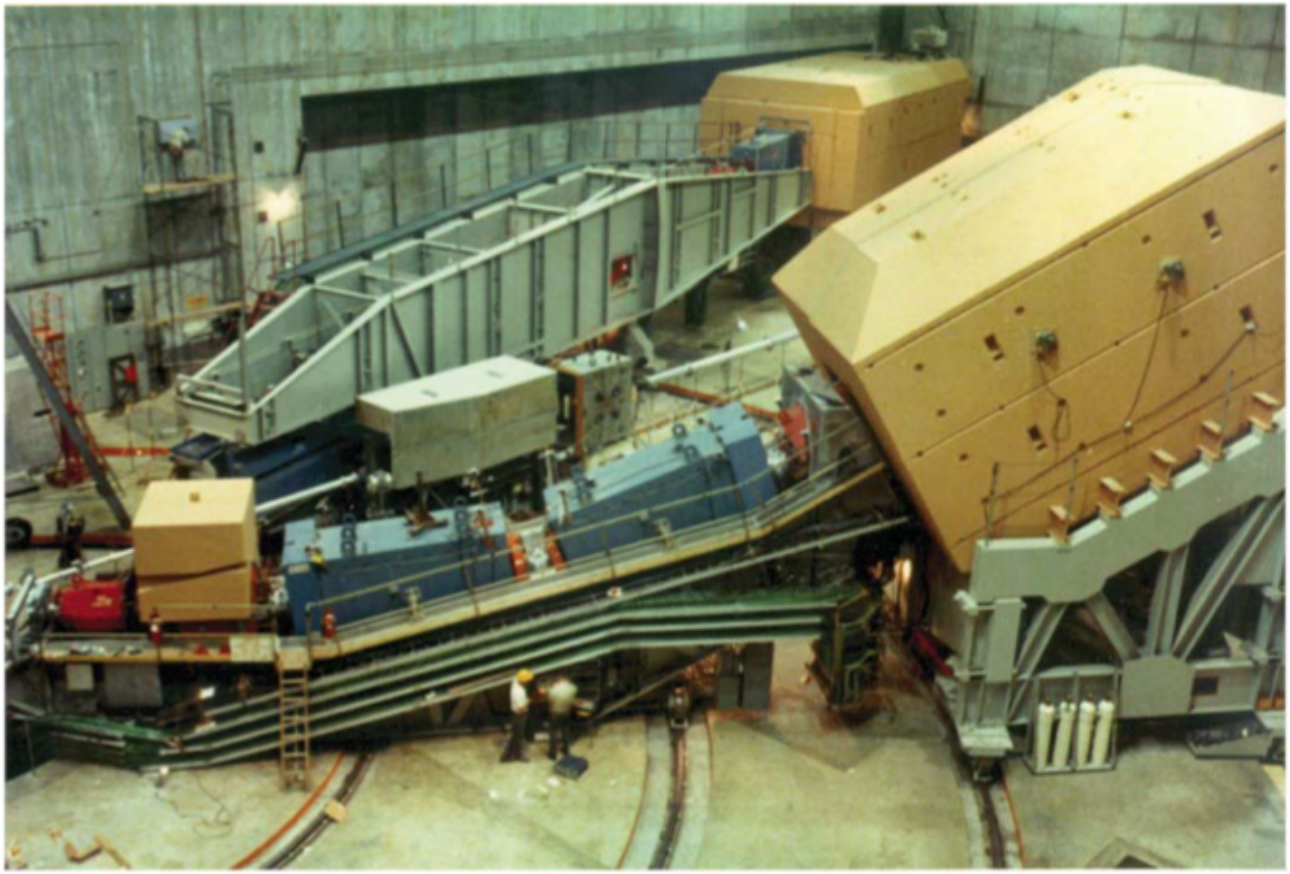

Elastic Scattering

Nuclear Physics Group Seminar

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Scattering Cross-Sections

- Describe the interaction (rate $\dot{N}$) of particles
- Consider a beam of cross-sectional area A and particle density n_A with particles having an average velocity v_a
- The particle flux describes the number of particles hitting the target per unit area and unit time

$$\Phi_a = n_a \cdot v_a = \frac{\dot{N}_a}{A}$$

- Consider a target of thickness d and particle density n_b
- The geometric cross-section is defined as

$$\sigma_b = \frac{\dot{N}}{\Phi_a \cdot N_b} = \frac{\dot{N}}{\dot{N}_a \cdot n_b \cdot d} \quad \text{with} \quad N_b = n_b \cdot A \cdot d$$

Luminosity

- This geometric scattering cross-section can be re-written

$$\sigma_b = \frac{\dot{N}}{\Phi_a \cdot N_b} = \frac{\dot{N}}{\mathcal{L}}$$

- Here the luminosity is defined by

$$\mathcal{L} = \Phi_a \cdot N_b = \dot{N}_a \cdot n_b \cdot d = n_a \cdot v_a \cdot N_b$$

Luminosity

- Luminosity for colliding beams with N bunches is given by

$$\mathcal{L} = \frac{N_a \cdot N_b \cdot N \cdot v \,/\, C}{4\pi \, \sigma_x \sigma_y}$$

with

– beam velocities v

– collider ring circumference C

– beam cross-section at collision point $4\pi \, \sigma_x \sigma_y$

Interaction Rate

- The interaction rate depends on the interaction potential described by an operator $\mathcal{H}$
- The operator $\mathcal{H}$ transforms the initial wave function ψ_i into the final wave function ψ_f
- The transition matrix element (transition probability amplitude) is given by

$$\mathcal{M} = \langle \psi_f | \mathcal{H} | \psi_i \rangle$$

Interaction Rate

- Consider a particle scattered into a volume V and momentum interval p' and $p' + \mathrm{d}p'$
- The interaction rate also depends on the number of final states available
- Ignoring spin, the number of final states is given by

$$\mathrm{d}n(p') = \frac{V \cdot 4\pi p'^2}{(2\pi\hbar)^3} \mathrm{d}p'$$

with $(2\pi\hbar)^3$ the volume of phase space occupied by each particle

- Energy and momentum are connected by

$$\mathrm{d}E' = v' \mathrm{d}p'$$

Interaction Rate

- The density of final states in the energy interval $\mathrm{d}E'$ is given by

$$\varrho(E') = \frac{\mathrm{d}n(E')}{\mathrm{d}E'} = \frac{V \cdot 4\pi p'^2}{v' \cdot (2\pi\hbar)^3}$$

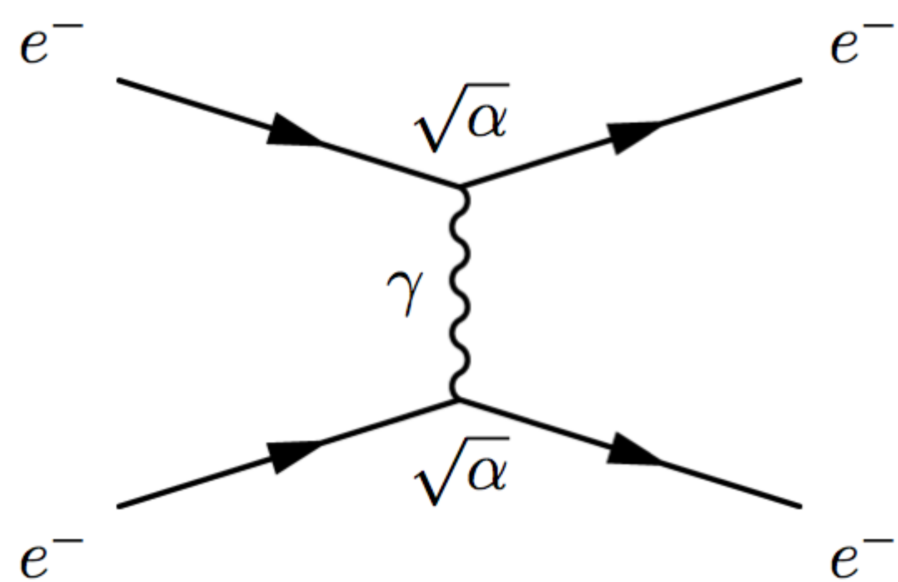
- Fermi's Golden Rule connects the interaction rate per target particle and per beam particle with the transition matrix element and the density of final states

$$W_{fi} = \frac{\dot{N}(E)}{N_a \cdot N_b}$$

$$= \frac{2\pi}{\hbar} \, |\, \mathcal{M} \,|^2 \cdot \varrho(E')$$

Feynman Diagrams

- Describe particle scattering through interacting currents
- A pictorial way of describing fermion and boson interactions
- Example electron-electron scattering



Transition Amplitudes

- Initial and final state particles have wave functions
- Vertices have dimensionless coupling constants
  - Electromagnetic interactions ⇒ $\sqrt{\alpha} \propto e$
  - Strong interactions ⇒ $\sqrt{\alpha} \propto \sqrt{\alpha_s}$
- Virtual particles have propagators
  - Virtual photon has propagator $\propto 1/q^2$
  - Virtual boson of mass m has propagator $\propto 1/(q^2 - m^2)$
- Transition amplitudes for electron-electron or electron-nucleon scattering have a virtual photon as propagator and two vertices, resulting in

$$\mathcal{M} \propto e^2 / q^2$$

with q^2 the 4-momentum transfer squared

Electron-Nucleon Scattering Kinematics

- Electron with incident 4-momentum $k = (E, 0, 0, E)$ and scattered 4-momentum $k' = (E', E' \sin\theta, 0, E' \cos\theta)$
- Nucleon with incident 4-momentum $P = (M, 0, 0, 0)$ and scattered 4-momentum $P' = (M + \nu, -E' \sin\theta, 0, E - E' \cos\theta)$
- Exchanged virtual photon with 4-momentum $q = k - k'$
- Invariant virtual photon mass squared is given by

$$q^2 = -Q^2 = -4\,E\,E' \cdot \sin^2(\theta/2)$$

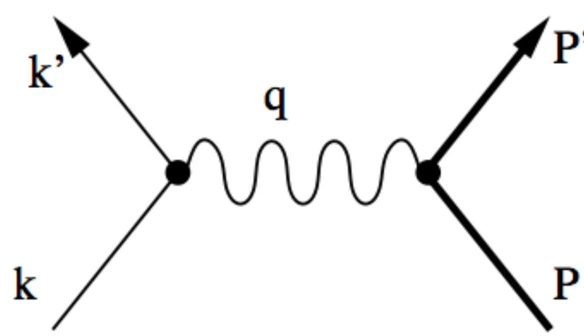
with θ the scattering angle in the lab frame

E the incident electron energy

E' the scattered electron energy

$\nu = E - E'$ the energy transfer

M the nucleon rest mass



Electron-Nucleon Scattering Kinematics

- Invariant mass squared of the final state nucleon is given by

$$W^2 = M^2 + 2M\nu - Q^2$$

- In elastic scattering $W = M$, which yields $Q^2 = 2M\nu$

Scattering Cross-Sections

- Rutherford cross-section for scattering of point-like particles (no recoil)

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{Z^2 \alpha^2 (\hbar c)^2}{4 E^2 \cdot \sin^4(\theta/2)} = \frac{4 Z^2 \alpha^2 (\hbar c)^2 E'^2}{Q^4}$$

- Mott cross-section accounting for spin $\boldsymbol{s}$ of electron

$$\left(\frac{d\sigma}{d\Omega}\right)^*_M = \left(\frac{d\sigma}{d\Omega}\right)_R \cdot (1 - \beta^2 \sin^2(\theta/2)) \qquad \text{with } \beta = v/c$$

- Helicity $h = \dfrac{\boldsymbol{s} \cdot \boldsymbol{p}}{|\boldsymbol{s}| \cdot |\boldsymbol{p}|}$ is conserved for relativistic particles

Nuclear Form Factors

- Experimental data only agree with Mott cross-section for very low $\boldsymbol{q}$
- The spatial extension of nuclei can be accounted for by form factors $F(\boldsymbol{q}^2)$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{exp}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{\mathrm{M}} \cdot \left|F(\boldsymbol{q}^2)\right|^2$$

- The form factors are the Fourier transform of the charge distribution $f(x)$

$$F(\boldsymbol{q}^2) = \int \mathrm{e}^{i\boldsymbol{q}x/\hbar} f(x)\, \mathrm{d}^3x$$

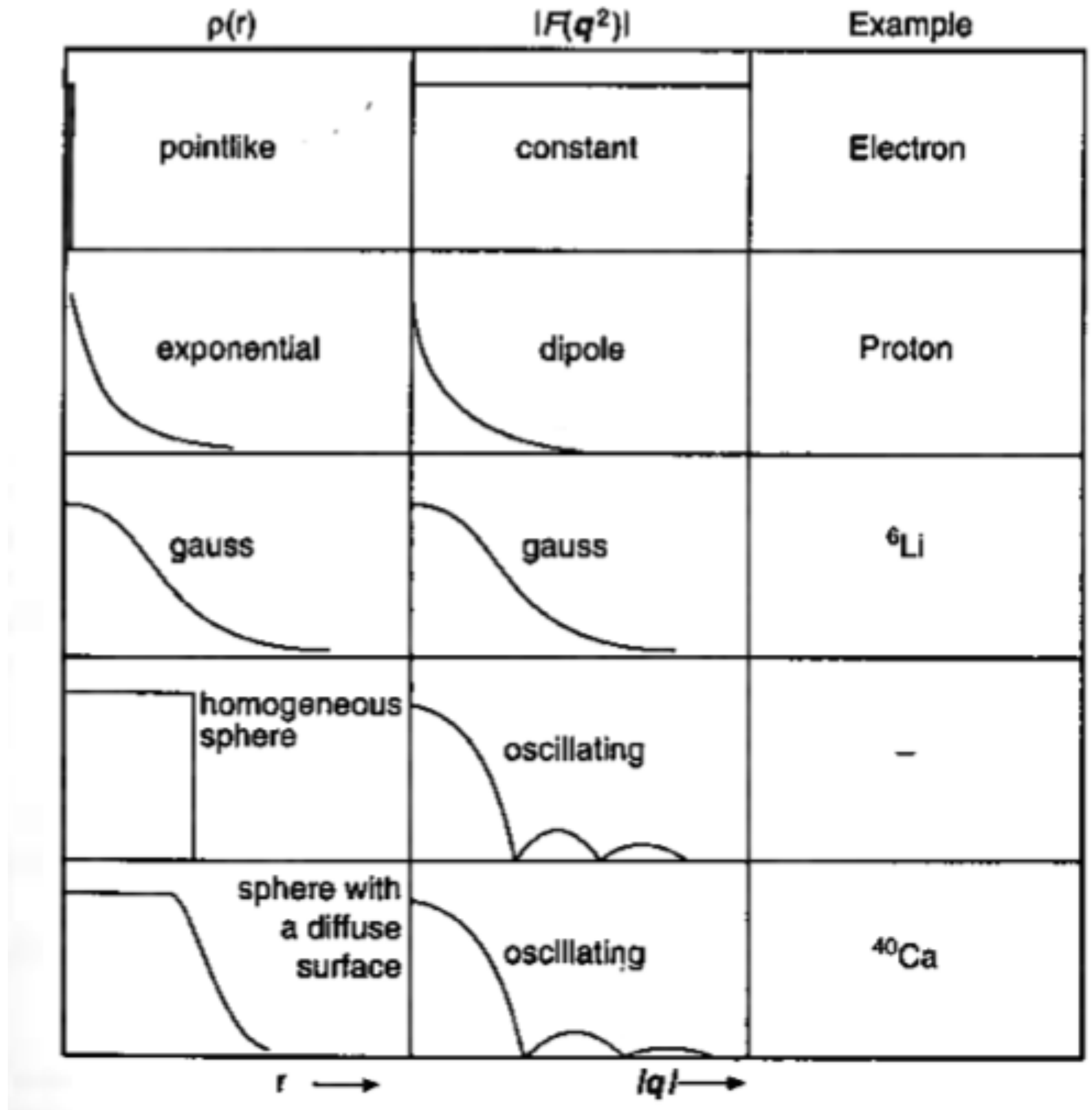
(Born approximation and no recoil)

Nuclear Form Factors

- For spherical symmetric cases the charge distribution only depends on the radius r

$$F(\boldsymbol{q}^2) = 4\pi \int \frac{\sin |\boldsymbol{q}| r/\hbar}{|\boldsymbol{q}| r/\hbar} f(r)\, r^2 \,\mathrm{d}r$$

Nuclear Form Factors



Nuclear Form Factors

Charge distribution $f(r)$		Form Factor $F(q^2)$	
point	$\delta(r)/4\pi$	1	constant
exponential	$(a^3/8\pi)\cdot\exp(-ar)$	$(1+q^2/a^2\hbar^2)^{-2}$	dipole
Gaussian	$(a^2/2\pi)^{3/2}\cdot\exp(-a^2r^2/2)$	$\exp(-q^2/2a^2\hbar^2)$	Gaussian
homogeneous sphere	$\begin{cases} 3/4\pi R^3 & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$	$3\,\alpha^{-3}(\sin\alpha - \alpha\cos\alpha)$ with $\alpha = \lvert q\rvert R/\hbar$	oscillating

Nucleon Form Factors

- Scattering of nucleons with mass of about 938 MeV requires recoil effects to be accounted for

$$\left(\frac{d\sigma}{d\Omega}\right)_M = \left(\frac{d\sigma}{d\Omega}\right)^*_M \cdot \frac{E'}{E}$$

$$= \frac{4 Z^2 \alpha^2 (\hbar c)^2 E'^2}{Q^4} \frac{E'}{E} \cdot (1 - \beta^2 \sin^2(\theta/2))$$

Nucleon Form Factors

- Electron interaction with the magnetic moment μ of the nucleon needs to be included also
- The magnetic moment of a charged, point-like spin-1/2 particle is given by

$$\mu = g \frac{e}{2M} \frac{\hbar}{2} \qquad \text{with} \quad g = 2$$

- We obtain for the scattering cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_M \cdot \left(1 + 2\,\tau \tan^2(\theta/2)\right)$$

$$\text{with} \quad \tau = \frac{Q^2}{4M^2c^2}$$

Nucleon Form Factors

- The magnetic moments of nucleon deviates from 2 because of the composite structure
- Measured values are

$$\mu_p = \frac{g_p}{2} \mu_N = +2.79 \cdot \mu_N$$ for the proton

$$\mu_n = \frac{g_n}{2} \mu_N = -1.91 \cdot \mu_N$$ for the neutron

with $\mu_N = \frac{e \hbar}{2 M_p}$

Nucleon Form Factors

- Charge and current distribution can be described by form factors as for nuclei
- Two form factors are needed to describe the charge and magnetic distributions
- The cross-section is given by the Rosenbluth formula

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{M}} \cdot \left(\frac{G_E^2(Q^2) + \tau\, G_M^2(Q^2)}{1+\tau} + 2\,\tau\, G_M^2(Q^2)\tan^2(\theta/2)\right)$$

- As $Q^2 \rightarrow 0$ also $\tau \rightarrow 0$ and only $G_E^2(0)$ remains above

with $\tau = \dfrac{Q^2}{4M^2c^2}$

Nucleon Form Factors

- For the form factors we expect in the limit $Q^2 \rightarrow 0$

$$G_E{}^p(0) = 1$$

$$G_M{}^p(0) = +2.79$$

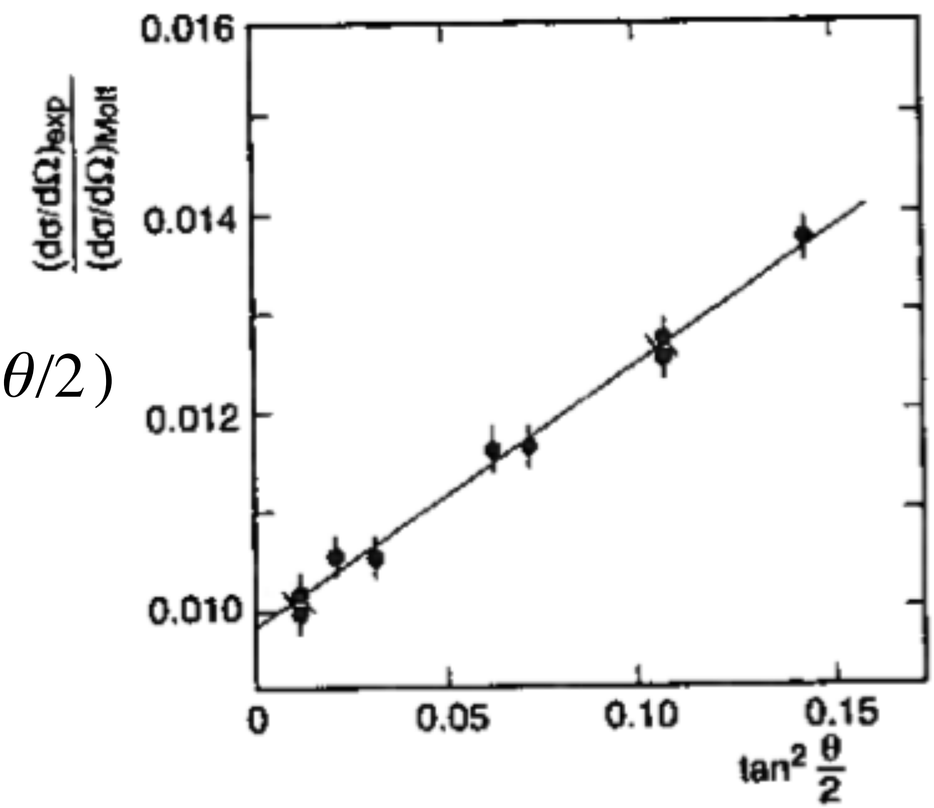
$$G_E{}^n(0) = 0$$

$$G_M{}^n(0) = -1.91$$

Measuring Nucleon Form Factors

- To determine the two form factors independently, we need to measure the cross section at fixed values of Q^2 and vary the beam energy (scattering angle)

$$\left(\frac{d\sigma}{d\Omega}\right) \Big/ \left(\frac{d\sigma}{d\Omega}\right)_M = \frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2(\theta/2)$$



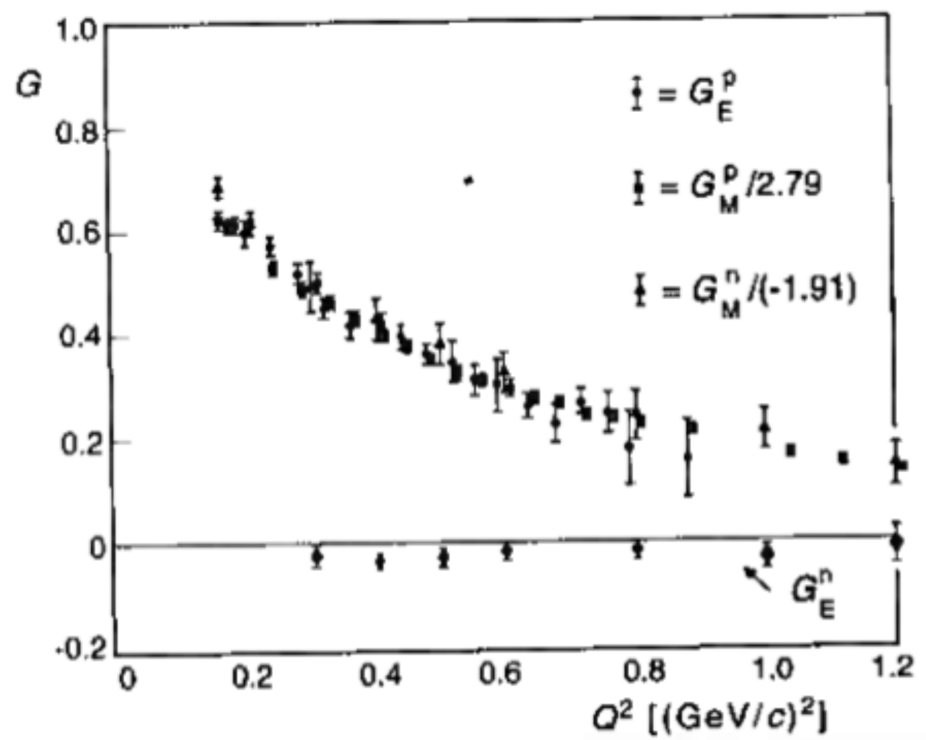
Measuring Nucleon Form Factors

- Result of measurements revealed dipole structure

$$G_E^{\,p}(Q^2) = \frac{G_M^{\,p}(Q^2)}{2.79} = \frac{G_M^{\,n}(Q^2)}{-1.91} = G^{dipole}(Q^2)$$

$$G^{dipole}(Q^2)$$

$$= (\,1 + Q^2 / (0.71\ \text{GeV}^2)\,)^{-2}$$



Nucleon Charge Distribution

- Spatial extend of nucleon charge can be found from measured form factors
- Fourier transform only applicable at low Q^2
- Dipole form factor corresponds to a diffuse and exponentially falling charge distribution

$$\varrho(r) = \varrho(0)\, e^{-a r} \qquad \text{with} \quad a = 4.27\ \text{fm}^{-1}$$

- The yields for the proton charge radius

$$\sqrt{\langle r^2 \rangle_p} = 0.86\ \text{fm}$$