Nuclear Physics - Problem Set 2 - Solution

Problem 1)

a) According to our formula for the Fermi momentum, we have

$$p_F^p = \left(\frac{3\pi^2\hbar^3 Z}{V}\right)^{\frac{1}{3}} = \left(\frac{3\pi^2\hbar^3 Z}{\frac{4\pi}{3}R^3}\right)^{\frac{1}{3}} = \left(\frac{9\pi Z}{4}\right)^{\frac{1}{3}} \frac{\hbar}{R_p} = 251.41\,MeV/c \quad \Rightarrow \quad T_F^p = 33.68\,MeV$$

and $$p_F^n = \left(\frac{9\pi N}{4}\right)^{\frac{1}{3}} \frac{\hbar}{R_n} = 276.96\,MeV/c \quad \Rightarrow \quad T_F^n = 40.82\,MeV$$

for the *maximum* (Fermi) kinetic energies of each species. (These numbers turn out higher than the "canonical" 250 MeV/c and 33 MeV for neutrons because Uranium has a large p-n imbalance). The *average* Fermi energies are 3/5 of these values, so the overall Fermi energy is $92 \cdot 3/5 \cdot 33.68 = 1859$ MeV for the protons and $146 \cdot 3/5 \cdot 40.82 = 3576$ MeV for neutrons. The Coulomb energy of a charged sphere is $\frac{3}{20\pi\varepsilon_o}\frac{q^2}{R} = \frac{3e^2}{20\pi\varepsilon_o R}Z^2$ (see the solution to Homework problem set 1) which yields an additional 1075.4 MeV. The combined Coulomb and kinetic energy of the protons equals 2934 MeV (a bit smaller than the "Fermi energy" of the neutrons alone), yielding a total binding energy of ^{238}U of $B = 238V_o + 6510$ MeV. Of course we would have expected both energies to be the same, because otherwise the overall binding energy could be reduced by converting (through β^- decay) some neutrons to protons. However, the Fermi gas model does not account properly for the details of nuclear binding.

b) The experimental value for B is B = -1801.7 MeV and therefore the total potential energy should be -8312 MeV or $V_o = -34.9$ MeV per nucleon. That is substantially deeper than the "average" of -28 MeV mentioned in the lecture, which has to do with the fact that ^{238}U is a doubly even nucleus and that it has a larger average density than typical nuclei.

c) The least bound neutron is of course right at the "Fermi edge" and therefore has kinetic energy $T_F^n = 40.82$ MeV. Hence, its overall energy should be $T_F^n + V_o = +6$ MeV, which is greater than zero. One should expect that this neutron simply flies off! However, the actual separation energy is $E_s^n = m\left(^{238}U\right) - m\left(^{237}U\right) - m\left(^{1}n\right) = -6.1528$ MeV which makes a whole lot more sense. The difference comes about because we have to subtract from the 6 MeV the amount of energy it takes to squeeze the remaining 237 nucleons into a volume which is just 1/238 smaller. This may seem like a small difference, but since it affects the kinetic energy of all remaining 237 nucleons, it actually adds up to quite a lot - 15.3 MeV (64 keV per nucleon) for a "predicted" neutron separation energy of -9.3 MeV. This would bring the two results in much better agreement.

Problem 2)

All of the nuclei given are either one nucleon shy of or one nucleon above a closed shell. This single nucleon then determines spin, parity and magnetic moment in the extreme single-particle picture. For the magnetic moment, we use the formula

$$\frac{\mu}{\mu_N} = \left(g_l \pm \frac{g_s - g_l}{2l+1} \right) j \quad \text{where} \quad j = l \pm \frac{1}{2}$$

For ^{3}He, we have one neutron missing in the $1s_{1/2}$ shell, which gives spin J=1/2 and parity π=+. The magnetic moment is μ/μ_N=+1/2g_s=-1.91 (experiment: -2.13).
For ^{5}He, we have one extra neutron in the $1p_{3/2}$ shell, which gives spin J=3/2 and parity π=-. The magnetic moment is μ/μ_N=+3/2g_s/3=-1.91 (same as before).

^{15}N has a single hole in the proton $1p_{1/2}$ shell, i.e. J=1/2 and ℓ=1 (parity π=-.). For a proton, g_ℓ = 1 and we get μ/μ_N= (1 - 4.58/3)·1/2 = -0.2633.

For ^{15}O, we have one neutron missing in the $1p_{1/2}$ shell, which gives spin J=1/2 and parity π=-. The magnetic moment is μ/μ_N=-1/2g_s/3=+0.638 (see book).
For ^{17}F, we have one extra proton in the $1d_{5/2}$ shell, which gives spin J=5/2 and parity π=+. The magnetic moment is μ/μ_N=5/2(1+[g_s-1]/5)=+4.79.
For ^{41}Ca, we have one extra neutron in the $1f_{7/2}$ shell, which gives spin J=7/2 and parity π=-. The magnetic moment is μ/μ_N=+7/2g_s/7=-1.91 (same as always - you begin to see the pattern: a single neutron always makes μ/μ_N=-1.91 if j=l+1/2, no matter which angular momentum ℓ it has).
Finally, for ^{131}In, we have one proton missing in the $1g_{9/2}$ shell, which gives spin J=9/2 and parity π=+. The magnetic moment is μ/μ_N=9/2(1+[g_s-1]/9)=+6.79. Again, the pattern becomes obvious: count 2.79 for the spin of the proton and one unit for each unit of ℓ - but only if ℓ and s are parallel (j=l+1/2).