Nuclear Physics - Problem Set 4 - Solution

Problem 1)

The initial and final electron four momenta (ignoring the electron mass) are $p_e^\mu = (E_e, 0, 0, E_e)$ and $p_e'^\mu = (E'_e, E'_e \sin\theta_e, 0, E'_e \cos\theta_e)$, respectively (I chose the z-axis along the initial beam direction, and the x-axis in the direction of the outgoing electron). Calculating the 4-momentum transfer Q^2 gives

$$Q^2 = -\left[\left(E_e - E_e'\right)^2 - \left(\vec{p}_e - \vec{p}_e'\right)^2\right] = -\left[\left(E_e - E_e'\right)^2 - \left(-E'_e \sin\theta_e\right)^2 - \left(E_e - E'_e \cos\theta_e\right)^2\right]$$

$$= -\left[\left(E_e - E_e'\right)^2 - E'^2_e - E_e^2 + 2E_e E'_e \cos\theta_e\right] = -\left[-2E_e E'_e + 2E_e E'_e \cos\theta_e\right] =$$

$$2E_e E'_e\left(1 - \cos\theta_e\right) = 4E_e E_e' \sin^2\frac{\theta_e}{2}$$

Let's call the mass of the nucleus m_A. In the initial state, it has four momentum $p_A^\mu = (m_A, 0, 0, 0)$ (setting c=1). 4-momentum conservation gives me the final 4-momentum of the nucleus: $p_A'^\mu = p_A^\mu + p_e^\mu - p_e'^\mu$. The fact that the scattering is elastic requires that the invariant mass squared of this 4-momentum equals the mass of the nucleus, m_A:

$$m_A^2 = p_A'^2 = p_A^2 + \left(p_e^\mu - p_e'^\mu\right)^2 + 2p_A\left(p_e^\mu - p_e'^\mu\right)$$

$$= m_A^2 - Q^2 + 2m_A\left(E_e - E_e'\right)$$

The first expression in the last line is the initial nucleus 4-momentum squared, the second term is using the definition of Q^2 given in the problem, and the last is the scalar product between p_A and p_e-p_e' (which has only one term since the 3-momentum part of p_A is zero). It follows immediately that $Q^2 = 2m_A\left(E_e - E_e'\right)$ holds.

Setting both sides equal yields the answer for the final question:

$$4E_e E_e' \sin^2\frac{\theta_e}{2} = 2m_A\left(E_e - E_e'\right) \quad \Rightarrow$$

$$\left(4E_e \sin^2\frac{\theta_e}{2} + 2m_A\right)E_e' = 2m_A E_e \quad \Rightarrow$$

$$E_e' = \frac{E_e}{1 + \frac{2E_e}{m_A}\sin^2\frac{\theta_e}{2}} \quad \Rightarrow \quad Q^2 = \frac{4E_e^2 \sin^2\frac{\theta_e}{2}}{1 + \frac{2E_e}{m_A}\sin^2\frac{\theta_e}{2}}$$

Problem 2)

Using the numbers given, we can calculate the virtual photon kinematics for both cases (all energies in MeV, all momenta in MeV/c):

$\nu = \{259.615, 66.104\}; Q^2 = \{198386, 250237\}; |\vec{q}| = \sqrt{Q^2 + \nu^2} = \{515.544, 504.585\}$

Given some initial momentum $\mathbf{p_i}$ for the proton, the final state Boron nucleus will move with momentum - $\mathbf{p_i}$ and will have energy

$E_{boron} = \sqrt{M_{boron}^2 + \vec{p}_i^{\,2}} = \sqrt{10252.5^2 + 200^2} = 10254.497$. Momentum conservation requires $\vec{p}_{p,final} = \vec{q} - \vec{p}_{Boron,final} \Rightarrow |\vec{p}_{p,final}| = q \pm 200 = \{715.544, 304.585\}$ for the two cases where the initial proton moves along $\mathbf{q}$ and the Boron moves opposite or vice versa. Finally, the energy of the final proton must be

$E_p = M_{Carbon} + \nu - E_{boron} = 11174.862\text{-}10254.497\text{+}\nu = \{1179.980, 986.470\}$

All that remains to be shown is that these two energies are consistent with a proton of mass 938.27 MeV/c^2 moving with the two final momenta calculated earlier:

$M_p \overset{!}{=} \sqrt{E_p^2 - \vec{p}_{p,final}^{\,2}} = \{938.2699999999999, 938.2700000000002\}$ which is the expected result within the numerical uncertainties (I used Mathematica to calculate these numbers). Of course, since the magnitude of the q-vector is larger than 200 MeV in all cases, the final state proton will always move in the direction of that vector; in the second case, it just flips its direction.

Problem 3)

Using the formulae from lecture, problem 1 and the book I find

Q^2= 3.413 $(\text{GeV/c})^2$ and $G_E^p = G_{Dipole} = 0.0297$ and $G_M^p = 2.79\ G_{Dipole} = 0.0828$.

Plugging it all in yields

$d\sigma/d\Omega = 7.74 \cdot 10^{-36}$ cm^2 / sr = 7.74 pb (pico-barn) / sr.