Nuclear and Particle Physics
Lecture Participation Project

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Elastic scattering of electron off nucleons

Historically, elastic electron-nucleon scattering was the first process to study the spatial distribution of charge and magnetism carried by nucleon.
When an electron scatters elastically from a proton, it exchanges a virtual photon as shown in figure 1.1 with the initial and final four momenta of electron as

$$k^{\mu} = (E, 0, 0, E) = (E, \vec{k})$$
$$k^{'\mu} = (E', E' sin\theta, 0, E' cos\theta) = (E', \vec{k'})$$

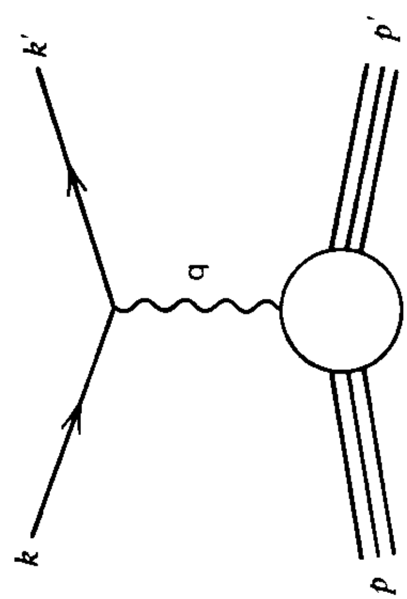

Figure 1: Sketch of elastic scattering of electron off a nucleus

The initial and final state four momenta of a nucleus are

$$p^{\mu} = (M, 0)$$
$$p^{'\mu} = (p^{\mu} + q^{\mu}) = (M + \nu, \vec{q})$$

where $\vec{q}$ is the spatial component of four momentum transfer

$$q^{\mu} = (E - E', -E' sin\theta, 0, E - E' cos\theta) = (E - E', \vec{k} - \vec{k'})$$

The momentum transfer q carried by virtual photon is constrained by 4 momentum vector conservation

$$k^{\mu} + p^{\mu} = k^{'\mu} + p^{'\mu}$$
$$k^{\mu} - k^{'\mu} = p^{'\mu} - p^{\mu} = q^{\mu}$$

Also, energy transfer to the recoiling nucleon

$$\nu = E - E' = |\vec{k}| - |\vec{k'}|$$

$$\vec{q} = \vec{k} - \vec{k'}$$

So,

$$q^\mu = (\nu, \vec{q}) = k^\nu - k'^\nu$$

The square of momentum transfer is Lorentz invariant that can be expressed in terms of incident energy E, final energy E' and scattering angle θ.

$$\begin{aligned} Q^2 &= -q^\mu q_\mu = -(k-k')^2 = \vec{q}^2 - \nu^2 \\ &= (E - E'cos\theta)^2 + E'^2 sin^2\theta - (E-E')^2 \\ &= E^2 - 2EE'cos\theta + E'^2 - E^2 - E'^2 + 2EE' \\ &= 2EE'(1-cos\theta) \\ &= 4EE'sin^2(\frac{\theta}{2}) \end{aligned} \tag{1}$$

In the laboratory frame, energy E' of scattered electron is:

$$E' = \frac{E}{1 + \frac{E}{Mc^2}(1-cos\theta)}$$

The recoil energy which is transferred to the target is given by the difference $E - E'$. In elastic scattering, a one-to-one relationship (above equation) exists between the scattering angle θ and energy E' of the scattered electron; which doesn't hold for inelastic scattering.

$$p^\mu = (M, 0)$$
$$p'^\mu = (p^\mu + q^\mu) = (M + \nu, \vec{q})$$

Total center of mass energy in final state

$$\begin{aligned} W^2 &= p'^\mu p'_\mu \\ &= (M+\nu)^2 - \vec{q}^2 \\ &= M^2 + 2M\nu + \nu^2 - \vec{q}^2 \end{aligned}$$

Since,

$$Q^2 = \vec{q}^2 - \nu^2 \tag{2}$$

Total invariant masses,

$$W^2 = M^2 + 2M\nu - Q^2 \tag{3}$$

For an elastic scattering, $W^2 = M^2$ So,

$$Q^2 = 2M\nu$$
$$\nu_{electron} = \frac{Q^2}{2M} \tag{4}$$

It relates a Lorentz invariant quantity Q^2 to another lorentz invariant quantity M.
Let us define a variable x defined as

$$\begin{aligned} x_{el} &= \frac{Q^2}{2M\nu_{el}} \\ &= \frac{Q^2}{2p^\mu q_\mu} \end{aligned} \tag{5}$$

For the following, the magnetic moment for protons and neutrons are respectively:

$$\mu_p = \frac{g_p}{2}\mu_N = +2.793\mu_N = (1+\kappa_p)\mu_N$$

and

$$\mu_n = \frac{g_n}{2}\mu_N = -1.913\mu_N = \kappa_n\mu_N$$

where the nuclear magneton

$$\mu_N = \frac{e\hbar}{2M_p} = 3.1525 \times 10^{-14} MeVT^{-1}$$

and the anomalous magnetic moments, $\kappa_p = 1.793$ and $\kappa_n = -1.913$, describe the deviation from the expected magnetic moment of a Dirac particle with charge $Q = 1$ (proton) or $Q = 0$ (neutron).

Form Factors:
The electromagnetic form factors describe the spatial distribution of electric charge and current inside the nucleon. The cross-section for the scattering of an electron off a nucleon is given by the Rosenbluth formula:

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha^2(\hbar c)^2 E'^2}{Q^4} \quad cos^2\frac{\theta}{2} \quad \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \quad + \quad 2\tau\ tan^2\frac{\theta}{2}G_M^2\right] \tag{6}$$

where, $G_E(Q^2)$ and $G_M(Q^2)$ are the electric and magnetic form factors depending on Q^2 with $\tau = \frac{Q^2}{4M^2}$, $\frac{E'}{E}$ is the recoil factor. The separation of contribution of G_E and G_M is usually performed by measuring the angular distribution of electron-proton elastic scattering at a fixed value of four-momentum transfer and plotting the cross section versus $tan^2\frac{\theta}{2}$.
For the limiting case $Q^2 \to 0$

$$G_E^p(Q^2 = 0) = 1$$
$$G_E^n(Q^2 = 0) = 0$$
$$G_M^p(Q^2 = 0) = 2.793$$
$$G_M^n(Q^2 = 0) = -1.913$$

The electric form factor of the proton and the magnetic form factors of both proton and neutron fall off similarly with Q^2:

$$G_E^p(Q^2) \approx \quad \frac{\mu_N\ G_M^p(Q^2)}{\mu_p} \approx \quad \frac{\mu_N\ G_M^n(Q^2)}{\mu_n} \approx \quad G^{dipole}(Q^2)$$

where

$$G^{(dipole)}(Q^2) = \left(1 + \frac{Q^2}{0.71(\frac{GeV}{c})^2}\right)^{-2} \tag{7}$$

For Dirac particle, $G_E = G_m = 1$
The interpretation of the form factors as the Fourier transform of the static charge distribution is only correct for small values of Q^2. The observed dipole form factor of Eq. 8 corresponds to the charge distribution which falls off exponentially .

$$\rho(r) \approx \frac{a^3\ e^{-ar}}{8\pi}$$
$$\Longrightarrow\ G_E(Q^2) \approx \left(\frac{1}{1+\frac{Q^2}{a^2}}\right) \tag{8}$$

with $a^2 = 0.71 GeV^2$ The contribution of G_E is the largest at small values of Q.