$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_i m_i \vec{r}_i'^{\,2}$$

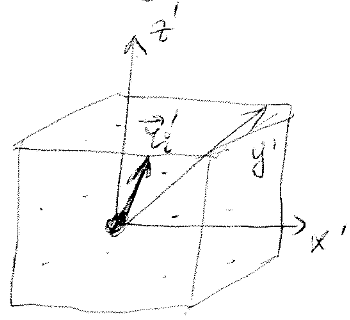
$$L = \cancel{\tfrac{1}{2}} \sum_i m_i \vec{r}_i \times \vec{v}_i = \cancel{\tfrac{1}{2}} \sum_i m_i (\vec{R} + \vec{r}_i') \times (\vec{V} + \vec{v}_i') =$$

$$= \underbrace{M \vec{R} \times \dot{\vec{R}}}_{\vec{L}_{c.m.}} + \underbrace{\sum_i m_i (\vec{r}_i' \times \vec{v}_i')}_{\vec{L}_{int}}$$

$$\frac{d\vec{L}_{c.m}}{dt} = M \vec{R} \times \ddot{\vec{R}} = R \times \sum \vec{F}_{ext.}$$

$$\frac{d\vec{L}_{int}}{dt} = \sum_i \vec{r}_i' \times \vec{F}_i^{ext}$$

If we have an object nailed down at a point, then we don't worry anymore about motion of c.m., but only look at rotations around that fixed point.



$$\frac{d_{space} \vec{V}}{dt} = \frac{d_{body} \vec{V}}{dt} + \vec{\omega} \times \vec{V}$$

$$\vec{L}_{int} = \sum_i m_i \vec{r}_i' \times (\vec{\omega} \times \vec{r}_i') =$$

(relative to the fixed point)

$$\vec{V}_{i\,space} = \vec{\omega} \times \vec{r}_i'$$

$$= \sum m_i \left[\vec{\omega} (\vec{r}_i')^2 - \vec{r}_i' (\vec{r}_i' \cdot \vec{\omega}) \right]$$

$$\begin{pmatrix} x_i'x_i' & x_i'y_i' & x_i'z_i' \\ y_i'x_i' & y_i'^2 & y_i'z_i' \\ z_i'x_i' & z_i'y_i' & z_i'^2 \end{pmatrix} \begin{pmatrix} \vec{\omega} \end{pmatrix}$$

$$\vec{r}_i' \circ \vec{r}_i' \qquad \text{dyadic product}$$

$$= \underbrace{\sum_i m_i \left(\vec{r}_i'^{\,2}\, \mathbb{1} - \vec{r}_i' \circ \vec{r}_i'\right)}_{= \mathbb{I} - \text{inertia tensor}} (\vec{\omega})$$

$\Rightarrow$ angular momentum is not necessarily $\propto$ to ω and may point to diff. direction than $\vec{\omega}$.
proportional

$$\mathbb{I}_{em} = \sum_i m_i \left(\vec{r}_i'^{\,2} \delta_{em} - r_e' r_m' \right)$$

$$(\mathbb{I})' = R\,(\mathbb{I})\,R^T \qquad R \text{ describes passive rotation into primed coordinate system}$$

$$\Rightarrow \vec{L}_{int} = \mathbb{I}\, \vec{\omega} \quad \text{can be evaluated in any coordinate system}$$

if we change coord. system $\underbrace{R\vec{L}_{int}}_{\text{changed}} = \underbrace{R\,\mathbb{I}\,R^T}_{\text{changed}} \underbrace{R\vec{\omega}}_{\text{changed}}$

If we have rotation around some axis then $\vec{\omega} = \omega \hat{n}$ ($\hat{n}$ $\perp$ dir. of axis)

$$\Rightarrow (\vec{L})_n = \hat{n} \cdot \vec{L} = \underbrace{(\hat{n}\, \mathbb{I}\, \hat{n})}_{I_n}\, \omega$$

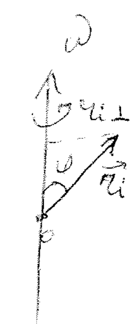
How can we use $\mathbb{I}$ to calculate T;

$$T = \frac{1}{2} \sum_i m_i \underbrace{\vec{V}_i' \cdot (\vec{\omega} \times \vec{r}_i')}_{\vec{\omega}(\vec{r}_i' \times \vec{V}_i')} = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot \mathbb{I} \cdot \vec{\omega} = \frac{1}{2} \omega^2 \hat{n}\, \mathbb{I}\, \hat{n} =$$

$$= \frac{1}{2} \omega^2 I_n \quad \llcorner \text{axis of rotation}$$

But also we can write

$$T = \frac{1}{2} \sum m_i (\vec{\omega} \times \vec{r}_i')(\vec{\omega} \times \vec{r}_i') = \frac{1}{2} \sum_i m_i \left(\omega_x r_{i\perp}\right)^2 = \frac{1}{2} \omega^2 \sum_i m_i r_{i\perp}^2$$



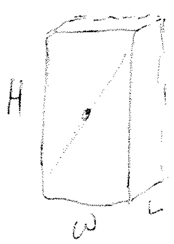
If we have homogeneous distribution

$$\Rightarrow \sum m_i \to \iiint \rho(\vec{r}')\, dx'\, dy'\, dz'$$

$$\Rightarrow \mathbb{T} = \iiint\limits_{\text{space}} \rho(\vec{r}\,')\,dx'dy'dz'\left(\vec{r}\,'^2\,\mathbb{1} - \vec{r}\,'\circ\vec{r}\,'\right) = \begin{pmatrix} y'^2+z'^2 & -x'y' & -x'z' \\ -y'x' & x'^2+z'^2 & -y'z' \\ -z'x' & -z'y' & x'^2+y'^2 \end{pmatrix}$$

$$I_x = \iiint \rho(x',y',z')\,dx'dy'dz'\,(y'^2+z'^2)$$

$$\mathbb{I}^T = \mathbb{I}$$



$$I_x = \rho L \iint dy'dz'\,(y'^2+z'^2)$$

$$\frac{y'^3}{3}\Big|_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \cdot H + \frac{z'^3}{3}\Big|_{-\frac{H}{2}}^{\frac{H}{2}} \cdot \omega = \rho L\left(\frac{\omega^3}{12}H + \frac{H^3}{12}\omega\right) =$$

$$= M\left(\frac{\omega^2}{12} + \frac{H^2}{12}\right)$$

$$\vec{r} = \vec{R} + \vec{r}\,'$$

$$\mathbb{I} = \sum m_i\left[(\vec{R}+\vec{r}_i\,')^2\,\mathbb{1} - (\vec{R}+\vec{r}_i\,')\circ(\vec{R}+\vec{r}_i\,')\right] =$$

(ralative to the point which is not center of mass)

$$= \vec{R}^2 M\mathbb{1} + \sum m_i \vec{r}_i\,'^2\,\mathbb{1} - M\vec{R}\circ\vec{R} - \sum m_i \vec{r}_i\,'\circ\vec{r}_i\,' =$$

$$= M\vec{R}^2\,\mathbb{1} - M\vec{R}\circ\vec{R} + \mathbb{I}_{c.m}$$ parallel axis theorem

Theorem There is a coordinate system, in which $\mathbb{I}$ has only diagonal elements.

$$R\mathbb{I}R^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Means we find three axes $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ such that

$$\mathbb{I}\vec{n}_1 = I_1 \vec{n}_1$$
$$\mathbb{I}\vec{n}_2 = I_2 \vec{n}_2$$
$$\mathbb{I}\vec{n}_3 = I_3 \vec{n}_3$$

we need to solve;

$$\det(\mathbb{I} - I \cdot \mathbb{1}) = 0$$

when we get $\underbrace{\vec{n}_1, \vec{n}_2, \vec{n}_3}_{\text{principal axes}} \Rightarrow N = \begin{pmatrix} \vec{n}_1 \end{pmatrix}\begin{pmatrix} \vec{n}_2 \end{pmatrix}\begin{pmatrix} \vec{n}_3 \end{pmatrix}$

$$N^{-1}\mathbb{I}N = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

We just need to prove that $\vec{n}_1, \vec{n}_2, \vec{n}_3$ eigenvectors are orthogonal. $\vec{n}_1 \cdot \vec{n}_2 = 0$, $\vec{n}_1 \vec{n}_3 = 0$, $\vec{n}_2 \vec{n}_3 = 0$.

$$\vec{n}_1 \underbrace{\mathbb{I}\vec{n}_2}_{\vec{n}_1 I_2 \vec{n}_2} = 0 \qquad (\vec{n}_1)^T \mathbb{I} = (\mathbb{I}^T n_1)^T = I_1 n_1$$

$$\Rightarrow \vec{n}_1 \mathbb{I} \vec{n}_2 = \vec{n}_1 I_2 \vec{n}_2 = \vec{n}_1 I_1 \vec{n}_2$$

$$\Rightarrow I_2 \vec{n}_1 \cdot \vec{n}_2 = I_1 \vec{n}_1 \cdot \vec{n}_2$$

$$\Rightarrow \text{if } I_1 \neq I_2 \quad \vec{n}_1 \cdot \vec{n}_2 = 0$$

if $I_1 = I_2$ $\Rightarrow$ find linear combination of $\vec{n}_1, \vec{n}_2$ such that it is perpendicular to $\vec{n}_2$

Principal axes: Symmetry axes. reflection axes