3-14

h = energy function

$\mathcal{L}(q_i, \dot{q}_i, t)$ all constaints "automatically" fulfilled

$i = 1, \ldots, k \leftarrow$ d.o.f.

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$h = \sum_{i=1}^{k} p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t)$$

Sometimes it is equal to total energy or not and/or conserved or not conserved

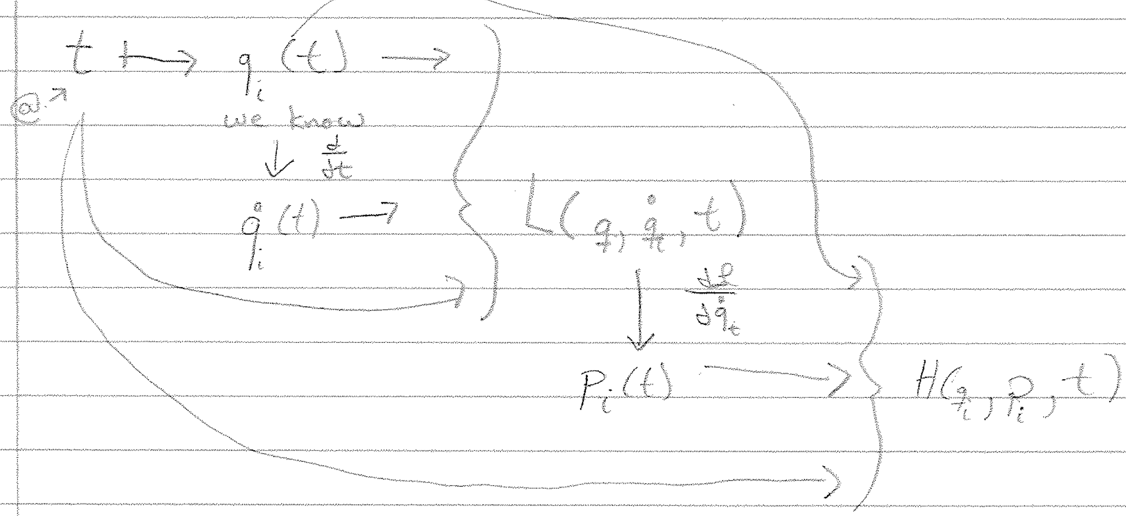
$h(q_i, \dot{q}_i, t)$

in order to see

$$dh = \sum_{i=1}^{k} \left[(d\,p_i)\dot{q}_i + \cancel{p_i\, d\dot{q}_i} \right] - \sum_{i=1}^{k} \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}}_{p_i} \cancel{d\dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial t} dt$$

$H = H(q_i, p_i, t)$

Think about following:



value(H) = value(h) (= E)

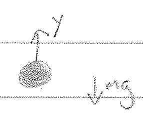
not always true

$$H(q_i, p_i, t) \Rightarrow \frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i$$

we know from ELE

$$\frac{d}{dt} p_i = \frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i \qquad \text{Hence} \qquad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

H is a generator of translation in time so that's why it's used in quantum ~~field~~ theory

Ex.: 

$T = \frac{1}{2} m \dot{y}^2 \qquad V = mgy$

$$\mathcal{L} = \frac{1}{2} m \dot{y}^2 - mgy$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = p_y = m\dot{y} \qquad \dot{y} = \frac{p_y}{m}$$

$$h = p_y \dot{y} - \mathcal{L} = \frac{1}{2} m \dot{y}^2 + mgy \qquad = E$$

$$H(y, p_y, t) = \frac{1}{2} m \frac{p_y^2}{m} + mgy = \frac{p_y^2}{2m} + mgy$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m} = \dot{y} \qquad \underbrace{\frac{\partial H}{\partial y} = mg = -\dot{p}_y \Rightarrow F_y = -mg}_{\text{new information}}$$

$$p_y = p_{y_0} - mgt \Rightarrow \dot{y} = \frac{p_{y_0}}{m} - gt \Rightarrow y(t) = y_0 + \frac{p_{0y}}{m} t - \frac{1}{2} g t^2$$

Now back to realization that $H(t)$

$$\frac{dH}{dt} = \left(\sum_i \underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i} \dot{q}_i + \underbrace{\frac{\partial H}{\partial p_i}}_{\dot{q}_i} \dot{p}_i \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

if H does not explicitly depend on t; conserved

* p_i becomes canonical momenta

Assume we can write $\mathcal{L}$ as

$$\mathcal{L} = \underbrace{\frac{1}{2} (\dot{\vec{q}})^T \underset{K \times K}{(\mathbb{T})} (\dot{\vec{q}})}_{\frac{1}{2}\sum_{i,j} \dot{q}_i \dot{q}_j T_{ij}(q,t)} + \underbrace{(\dot{\vec{q}})^T (\vec{a})}_{\sum_i \dot{q}_i a_i(\vec{q},t)} + \mathcal{L}_0(\vec{q},t) \qquad \vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}$$

$$(\mathbb{T}) = (\mathbb{T})(\vec{q},t) \qquad (\mathbb{T})^T = (\mathbb{T})$$

$$p_\ell = \frac{\partial \mathcal{L}}{\partial \dot{q}_\ell} = \sum_i \dot{q}_i T_{i\ell} + a_\ell = \left((\dot{\vec{q}})^T (\mathbb{T}) \right)_\ell + (\vec{a})^T_\ell \qquad (\vec{p})^T = (p_1 \ldots p_K) = (\dot{\vec{q}})^T (\mathbb{T}) + (\vec{a})^T$$

$$h = (\vec{p})^T (\dot{\vec{q}}) - \mathcal{L} = (\dot{\vec{q}})^T (\mathbb{T}) (\dot{\vec{q}}) + (\vec{a})^T (\dot{\vec{q}}) - \mathcal{L}$$

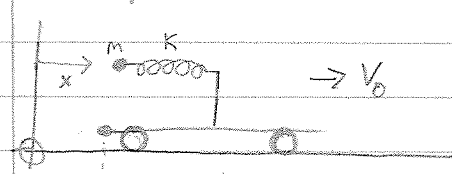
$$= \frac{1}{2} (\dot{\vec{q}})^T (\mathbb{T}) (\dot{\vec{q}}) - \mathcal{L}_0(\vec{q},t)$$

$$\boxed{(\dot{\vec{q}})^T = \left[(\vec{p})^T - (\vec{a})^T \right] (\mathbb{T})^{-1}}$$

so

$$\mathcal{H} = \frac{1}{2} (\vec{p} - \vec{a})^T (\mathbb{T})^{-1} (\vec{p} - \vec{a}) - \mathcal{L}_0 = H(\vec{p}, \vec{q}, t)$$

Example from book



$T = \frac{1}{2} m \dot{x}^2$

$x_{car} = v_0 t$ $\qquad V = \frac{1}{2} k (x - v_0 t)^2$

$\Pi = (m) \qquad \mathcal{L}_0 = -\frac{1}{2} k (x - v_0 t)^2$

$p = \dot{x} m \qquad \vec{a} = 0$

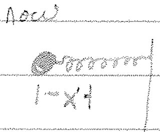
$H = \frac{1}{2m} p^2 + \frac{1}{2} k (x - v_0 t)^2$

$\dot{x} = \frac{p}{m} \qquad \dot{p} = \frac{-\partial H}{\partial x} = -k(x - v_0 t)$

There is a time dependance so E is not conserved

$p = \dot{x} m \qquad \dot{p} = \ddot{x} m \Rightarrow \ddot{x} = -\frac{k}{m}(x - v_0 t)$

now



$V = \frac{1}{2} k x'^2$

$T = \frac{1}{2} m (\dot{x}' + v_0)^2$

$\mathcal{L} = \frac{1}{2} m \dot{x}'^2 + m v_0 \dot{x}' + \left(\frac{1}{2} m v_0^2 - \frac{1}{2} k x'^2\right) \qquad (\pi) = (m); (\vec{a}) = m v_0$

$p' = m\dot{x}' + m v_0$

$H = \frac{1}{2}(p' - m v_0)^2 \frac{1}{m} - \frac{m}{2} v_0^2 + \frac{1}{2} k x'^2$ $\quad$ * is not E but conserved

$\to \frac{\partial H}{\partial p'} = \frac{p'}{m} - v_0 = \dot{x}' \qquad \dot{p}' = -\frac{\partial H}{\partial x'} = -kx'$

or reform !!

$\ddot{x}' = \frac{\dot{p}'}{m} = -\frac{k}{m} x'$ $\quad$ ($m\ddot{x}'$)