Reminder

Consider two vector spaces \( \mathbb{V} \) and \( \mathbb{U} \). We want to define the “direct product space” \( \mathbb{V} \otimes \mathbb{U} \).

We take any two basis elements from \( \mathbb{V} \) and \( \mathbb{U} \), \( |V_i, U_j > = |V_i > \otimes |U_j > \), as a basis state of \( \mathbb{V} \otimes \mathbb{U} \).

Assume \( \mathbb{V} \) has 2 dimensions and \( \mathbb{U} \) has 3 dimensions.

\[
|V_i > = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
|U_i > = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

Then the basis elements will be,

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

etc. (6 in total)

General state,

\[
|\psi_{\mathbb{V} \otimes \mathbb{U}} > = \sum_{ij} \alpha_{ij} |V_i > \otimes |U_j >
\]

in general, can not be written as product of just two states from \( \mathbb{V} \) and \( \mathbb{U} \):

\[
|\psi_{\mathbb{V} \otimes \mathbb{U}} > \neq |\psi_{\mathbb{V}} > \otimes |\psi_{\mathbb{U}} >
\]

Example: Considering the combination

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]
\]

there is no way to write it as just a single product \( \begin{pmatrix} a \nonumber \\ b \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ ae \\ bc \\ bd \\ be \end{pmatrix} \) of states from \( \mathbb{V} \) and \( \mathbb{U} \).
Another Example:

Let $\mathbb{V} = \mathbb{V}_l$ be the space of solutions with fixed angular momentum $l$ to the radial part of the Schrödinger equation. Let $\mathbb{U}$ be the space of solutions to the angular Schrödinger equation $|l, m >$ for the same angular momentum $l$, with magnetic quantum number $-l \leq m \leq l$.

For free states, $\text{Dim } \mathbb{V} = \mathbb{R}$; for bound states $\text{Dim } \mathbb{V} = \mathbb{Z}$. $\text{Dim } \mathbb{U} = 2l + 1$. Basis vectors of $\mathbb{V} \otimes \mathbb{U}$ are

$$|\psi_{Elm} > = |R_{El} > \otimes |l, m >$$

$$< \vec{r}'|\psi_{Elm} > = R_{El}(r)Y_l^m(\theta, \varphi)$$

Assume a Hamiltonian $H$ that commutes with $L^2$

$$[H, L^2] = 0$$

Here $l \rightarrow l'$ transition is not possible (if $l \neq l'$). Therefore, all solutions of the Schrödinger equation can be chosen to be Eigenvectors to $L^2$ with fixed $l$ (or linear combinations thereof).

Any wave function fulfilling the (time-dependent) Schrödinger equation in this subspace with fixed $l$ can be written as

$$|\psi_l > (t) = \int \sum_{m} a(E, m) |R_{El} > \otimes |l, m > e^{-iEt/\hbar} dE$$

Since $\mathbb{U}$ can be thought of consisting of column vectors of $2l + 1$ complex numbers, the most general vector in $\mathbb{V} \otimes \mathbb{U}$ is of the form

$$|\psi > = \begin{pmatrix}
R_l(r) \\
R_{l-1}(r) \\
\vdots \\
R_{-l}(r)
\end{pmatrix}$$

For the previous wave function $|\psi_l > (t)$, $R_l(r) = \int a(E, m = l) R_{El}(r)e^{-iEt/\hbar} dE$ etc.

In this interpretation, the basis elements look as follows:

$$\begin{pmatrix}
0 \\
0 \\
\vdots \\
R_{El}(r)
\end{pmatrix}$$

e tc.
Previously, we have shown that the angular momentum operators $J^2, J_z$ allow not only integer, but also half-integer values for the quantum numbers $j, m$ where the eigenvalues of $J^2$ are $j(j + 1)\hbar$ and the eigenvalues for $J_z$ are $m\hbar, -j \leq m \leq j$ (in integer increments).

For each value of $j$ we define a $(2j+1)$ – dimensional subspace with basis $|j, m \rangle$. How do we interpret the half-integer values of $j$? It turns out that in addition to orbital angular momentum $\mathbf{L}$ (which can only have integer values for $l$), there is also an intrinsic property of each (elementary) particle called spin $s$ (somewhat akin to rotation of a body around its own axis). In fact, any Hamiltonian that is consistent with special relativity must commute with this quantity for elementary particles: $[H, S^2] = 0$. Other than charge, the only absolute invariants for elementary particles are their mass $m$ and spin $s$, which therefore serve to define them.

<table>
<thead>
<tr>
<th>Particle (elementary ones are bold)</th>
<th>Spin $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higgs, $\pi, K, ^4\text{He}$</td>
<td>0</td>
</tr>
<tr>
<td>$\nu, \mu, e, \text{quarks}, ^3\text{He}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\gamma, W, Z, \rho$</td>
<td>1</td>
</tr>
</tbody>
</table>

Each such particle therefore must “live in” a sub space of defined spin $s = 0, +1/2, +1, +3/2, \ldots$ with possible basis states $|m_s \rangle, -s \leq m_s \leq s$.

Similar to orbital angular momentum operators, here we have spin operators $S_x, S_y, S_z, S^2, S_+, S_-$ which represent (infinitesimal) rotations in this new space and fulfill all the usual commutator rules as well as relationships like

$$S_z S_+ |m_s \rangle = \hbar(m_s + 1)S_+ |m_s \rangle$$

In general, a particle with spin $s$ must then be represented in the product space of its spatial coordinates, $\mathbb{V}$ with $\text{Dim } \mathbb{V} = \mathbb{R}^3$, and its “spin coordinates”, $\mathbb{U}$, with $\text{Dim } \mathbb{U} = 2s + 1$.

The basis states in $\mathbb{V} \otimes \mathbb{U}$ are given by $|\alpha, m \rangle = |\alpha \rangle \otimes |m_s \rangle$; $m_s = +s, \ldots, -s$ ($\alpha$ represents any quantum numbers describing the basis states in spatial coordinates):

$$< \mathbf{r} | \alpha, m_s \rangle = R_{\alpha} (\mathbf{r}) \otimes |m_s \rangle$$

Most general state = $$\begin{pmatrix} R_{s}(\mathbf{r}) \\ R_{s-1}(\mathbf{r}) \\ \vdots \\ R_{-s}(\mathbf{r}) \end{pmatrix}$$
If \( H = H_{\text{spatial}} + H_{\text{spin}} \), \([H_{\text{spatial}}, \vec{S}] = 0\) and \( H_{\text{spin}} \) acts only on spin degrees of freedom, then all eigenstates of \( H \) can be chosen in the form \(|\alpha > \otimes |m_s >\).

The simplest non-trivial case is spin-1/2: 

\( S = \frac{1}{2} \) which yields \( \text{Dim} \ U = 2 \), so \( U = \mathbb{C}^2 \) with basis states \((\frac{1}{0})\), \((0\frac{1}{1})\)

which are eigenfunctions to \( S_z \) with magnetic quantum numbers \( m_s = \pm \frac{1}{2} \):

\[
S_z \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{(*)}
\]

General \(|\psi > \in U \Rightarrow (\alpha \beta) ; \alpha, \beta \in \mathbb{C}\)

If we normalize the vector, then \(|\alpha|^2 + |\beta|^2 = 1 \Rightarrow\) we can write \(|\alpha| = \cos \gamma, |\beta| = \sin \gamma\)

\[
\Rightarrow |\psi > = \begin{pmatrix} \cos \gamma & e^{i\delta_\alpha} \\ \sin \gamma & e^{i\delta_\beta} \end{pmatrix}
\]

\[
= e^{i\frac{\delta_\alpha+\gamma_\beta}{2}} \begin{pmatrix} \cos \gamma & e^{\frac{i}{2}(\delta_\alpha-\gamma_\beta)} \\ \sin \gamma & e^{-\frac{i}{2}(\delta_\alpha-\gamma_\beta)} \end{pmatrix}
\]

\[
= e^{i\frac{\gamma_\alpha+\gamma_\beta}{2}} \begin{pmatrix} \cos \gamma & e^{i\Delta_\gamma} \\ \sin \gamma & e^{-\frac{i}{2}\Delta_\gamma} \end{pmatrix}
\]

Any operator in this vector space must be represented by a 2x2 matrix:

\[
\hat{O} = \begin{pmatrix} O_{1,1} & O_{1,-1} \\ O_{-1,1} & O_{-1,-1} \end{pmatrix}
\]

which can be expressed as a linear combination of 4 “basis” matrices:

\[
\hat{O} = \sum_{i=0,x,y,z} \theta_i \sigma_i
\]

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

In particular, the three components of the spin vector operator are

\[
S_i = \frac{\hbar}{2} \sigma_i \ (i = x,y,z). \text{ This can be proven as follows:}
\]

\[
S_z = \frac{\hbar}{2} \sigma_z \text{ follows simply from the definition of the basis, Eq. (**)}
\]
Similarly, it must be true that \( S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) since \( S_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) according to our results for arbitrary \( j \), and \( S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then all we need is that \( S_x = \frac{1}{2} (S_+ + S_-) \) and \( S_y = \frac{1}{2i} (S_+ - S_-) \).

Finally, \( S^2 = \frac{3}{4} \hbar^2 \mathbb{1} = s(s+1) \hbar^2 \sigma_0 \). This does not give anything new. It commutes with all possible operators as it must.

Some properties of the Pauli matrices:

\[
\sigma_i \sigma_j = i \sum_k \varepsilon_{ijk} \sigma_k + \delta_{ij} \sigma_0
\]

\[
\sigma_i \sigma_j = -\sigma_j \sigma_i \implies \text{The Pauli matrices anti-commute.}
\]

\[
[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k
\]

Eigen functions of \( S_z \): \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \)

\[
S_x \implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]

\[
S_y \implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}
\]

As shown above, up to a constant phase (irrelevant), any properly normalized state can be written like

\[
\begin{pmatrix}
\cos \gamma & e^{-\frac{i}{2} \Delta} \\
\sin \gamma & e^{\frac{i}{2} \Delta}
\end{pmatrix}
\]

A rotation around the \( \hat{n} \) is given by \( e^{-\frac{i \theta \hat{n} \cdot \hat{\sigma}}{\hbar}} = \mathbb{1} - i \theta \frac{\hat{n} \cdot \hat{\sigma}}{\hbar} + (-i \theta)^2 \frac{\hat{n} \cdot \hat{\sigma}}{\hbar} + \cdots \)

We know that \( \frac{\hat{\sigma}}{\hbar} = \frac{\hat{n} \cdot \hat{\sigma}}{\hbar} \)

\[
e^{-\frac{i \theta \hat{n} \cdot \hat{\sigma}}{\hbar}} = \mathbb{1} - i \theta \frac{\hat{n} \cdot \hat{\sigma}}{\hbar} + \frac{(-i \theta)^2}{2} (\hat{n} \cdot \hat{\sigma})^2 + \cdots
\]

\[
(\hat{n} \cdot \hat{\sigma})^2 = (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y + \hat{n}_z \sigma_z)(\hat{n}_x \sigma_x + \hat{n}_y \sigma_y + \hat{n}_z \sigma_z)
\]

\[
= (\hat{n}_x^2 \sigma_x^2 + \hat{n}_y^2 \sigma_y^2 + \hat{n}_z^2 \sigma_z^2) = \mathbb{1}
\]
\[
\begin{align*}
&= \sum_{\text{odd } n} \frac{1}{n!}(\frac{-i\theta}{2})^n \hat{n} \cdot \vec{\sigma} + \sum_{\text{even } n} \frac{1}{n!}(\frac{-i\theta}{2})^n \mathbb{1} \\
&= -i \sin \frac{\theta}{2} \hat{n} \cdot \vec{\sigma} + \cos \frac{\theta}{2} \mathbb{1} \\
&= \begin{pmatrix}
\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \hat{n}_z & -i \sin \frac{\theta}{2} \hat{n}_x - \sin \frac{\theta}{2} \hat{n}_y \\
-i \sin \frac{\theta}{2} \hat{n}_x + \sin \frac{\theta}{2} \hat{n}_y & \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \hat{n}_z
\end{pmatrix}
\end{align*}
\]

Rotation $\phi$ around z axis:

\[
\begin{pmatrix}
\cos \frac{\phi}{2} & 0 \\
0 & \cos \frac{\phi}{2}
\end{pmatrix}
\]

Rotation around z axis changes the relative phase of the two components of a spinor.

Rotation around y-axis:

\[
\begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\]

Combination (Euler angles) corresponds to rotating the spinor pointing in $+z$-direction to the direction given by the spherical coordinates $\theta, \phi$:

\[
\begin{pmatrix}
\cos \frac{\phi}{2} & 0 \\
0 & \cos \frac{\phi}{2}
\end{pmatrix}\begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix}
= \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i\phi/2} \\
\sin \frac{\theta}{2} e^{i\phi/2}
\end{pmatrix}
\]

This is exactly the form of the **most general** state possible if we identify $\theta/2 = \gamma$ and $\phi = \Delta \delta$! This means that for each possible state of a spin-$1/2$ particle, there is (exactly) one direction in space (given by the spherical coordinates $\theta, \phi$) such that it is in the eigenstate with $m_s = +1/2$ of the spin operator pointing in that direction, $\hat{n} \cdot \vec{S}$. Of course, for all **other** directions (except $-\hat{n}$), the state is **not** in an eigenstate of the corresponding spin operator, and therefore will have a statistical uncertainty for any measurement of the spin component along that direction.
Force Due to a magnetic field $B$ on a length $s$ of wire carrying current $I$:

$$F = BI_s$$

Torque on square loop with side length $s$:

$$\tau = 2s/2 BI_s \sin \theta$$

Magnetic moment:

$$\mu = Is^2$$

$$\vec{\mu} = Ia \hat{n}$$

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Work $dW = \tau d\theta$

Let initial orientation be at $\theta = 90$°

Work done:

$$\text{Work done} = \mu B \int_{90}^{\theta_{\text{final}}} \sin \theta \ d\theta$$

Potential energy stored in the loop, $V_{\text{pot}} = -\mu B \cos \theta_{\text{final}}$

Magnetic dipole moment of a single charge $q$ orbiting at fixed radius $r$ with velocity $v$:

$$\mu = \frac{qv}{2\pi r} r^2 = \frac{qv^2}{2} = \frac{q}{2mc} L$$

Interaction Hamiltonian is given by,

$$H_{\text{int}} = -\vec{\mu} \cdot \vec{B}$$

$$= -\frac{q}{2mc} \vec{j} \cdot \vec{B}$$

Electron Spin:

$$H_{\text{int}} = -g \frac{q}{2mc} \vec{S} \cdot \vec{B}$$

$$= g \frac{eh}{2mc} \hat{1} \cdot \vec{B}$$

Here $\mu_B = \frac{eh}{2mc}$ and $g = 2(1.00116)$

$$= -\gamma \vec{S} \cdot \vec{B}$$

In general Hamiltonian can have...

$$|\psi >_{\text{spatial}} \otimes \chi ; \chi = \left( \begin{array}{c} a \\ b \end{array} \right)$$

$$H = \frac{\vec{p}^2}{2m} + g \mu_B \frac{1}{2} \vec{\sigma} \cdot \vec{B}$$
\[ |\psi > (t=0) \Rightarrow |\psi > (t) = e^{-i\frac{\hat{H}t}{\hbar}} |\psi > (t=0) \]

Rotation around axis of \( \vec{B} \) by an angle of \( g \mu_B \frac{Bt}{\hbar} \)

\[ |\psi > (t) = e^{-i g \mu_B \vec{B} \cdot \vec{\sigma} \frac{t}{2 \hbar}} \]

Here
\[ g_{\text{proton}} = 2(2.79) \]
\[ g_{\text{neutron}} = 2(-1.91) \]

If the magnetic field is inhomogeneous;

Force = \(-\nabla V_{\text{pot}} = \vec{\mu} \cdot \nabla \vec{B} \)

Consequence: Stern-Gerlach apparatus which can measure the angular momentum (spin) component along the z-direction determined by the field direction.